

NECESSARY AND SUFFICIENT CONDITIONS FOR THE STRONG LOCAL MINIMALITY OF C^1 EXTREMALS ON A CLASS OF NON-SMOOTH DOMAINS

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ABSTRACT. Motivated by applications in materials science, a set of quasiconvexity at the boundary conditions is introduced for domains that are locally diffeomorphic to polytopes. These conditions are shown to be necessary for strong local minimisers in the vectorial Calculus of Variations and a quasiconvexity-based sufficiency theorem is established for C^1 extremals defined on this class of non-smooth domains. The sufficiency result presented here thus extends the seminal theorem by Grabovsky & Mengesha (2009), where smoothness assumptions are made on the boundary.

KEYWORDS: quasiconvexity at the boundary, polytopes, non-smooth domains, sufficient conditions, strong local minimisers

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1. INTRODUCTION

The problem of finding necessary and sufficient conditions for a map to be a strong local minimiser of a given functional is a fundamental question in the Calculus of Variations. The scalar case was first solved by Weierstrass and later, Hestenes [Hes48] extended these results by considering minimisers that allow more than one independent variable.

Regarding the vectorial case, the problem has received considerable attention over the past decades. In [Zha92] Zhang proved a sufficiency theorem in which the strong quasiconvexity of the integrand is used to prove that smooth solutions to the weak Euler-Lagrange equations are spatially-local minimisers. In other words, this means that minimality can be obtained when considering sufficiently small domains.

Ball conjectured in [Bal98] that a natural way to extend the Weierstrass theory to the vectorial case had to be based on the conditions of quasiconvexity at the interior and at the boundary, the latter having been introduced in [BM84a] and shown to be a necessary condition for strong local minima (see also [Kru13, KKK14, MS98] for other works related to the notion of quasiconvexity at the boundary). More precisely, Ball's conjecture states that under suitable quasiconvexity conditions, if a map is sufficiently smooth, satisfies the weak Euler-Lagrange equations and the strict positivity of the second variation, then it is a strong local minimiser.

Further achievements for the vectorial case were obtained in [Tah01]. In addition, Kristensen and Taheri showed that, under strong quasiconvexity and p -growth conditions, $W^{1,p}$ -local minimisers are *partially regular*, i.e., of class $C^{1,\alpha}$ almost everywhere in their domain. Their result extended the partial regularity that Evans had established for the first time for global minimisers under similar assumptions on the integrand [Eva86]. Furthermore, and in contrast with the previous achievements on the vectorial problem, Kristensen and Taheri modified the remarkable example by Müller and Šverák in [MŠ03] to construct a strongly quasiconvex integrand with strictly positive second variation, such that the corresponding weak Euler-Lagrange equations admit Lipschitz solutions that are not $W^{1,p}$ -local minimisers [KT03]. This made it clear that Lipschitz regularity of the extremal is not enough to ensure strong local minimality. Székelyhidi extended further the example of Kristensen and Taheri to the case of polyconvex integrands [Szé04].

However, it was only until the seminal paper [GM09] of Grabovsky and Mengesha that the conjecture of Ball was settled. They showed that, for domains of class C^1 , a strong version of the aforementioned quasiconvexity conditions is in fact sufficient for a C^1 map to be a strong local minimiser of an integral functional under specific growth and coercivity assumptions on the integrand.

The aim of the current work is to address the vectorial problem without the C^1 regularity assumption on the domain but instead allowing certain types of singularities of the boundary. More specifically, we establish a quasiconvexity-based sufficiency result for minimisers defined on domains that are locally diffeomorphic to a polytope (see Definition 2.6 for a precise definition).

Similarly to the work in [GM09], it is shown that when complemented with the satisfaction of the weak Euler-Lagrange equations and the (strong) positivity of the second variation, a strong version of the quasiconvexity in the interior and the suitably adapted conditions of quasiconvexity at the singular boundary are indeed sufficient for a C^1 map to be a strong local minimiser. The need to extend the Weierstrass theory to domains of this type arises naturally in view of applications. For example, models based on energy minimisation, and consequently the techniques of the vectorial Calculus of Variations, have been very successful in materials science where typical specimens are polyhedral. Indeed, the work presented here has been largely motivated by that in [BK16] where, in a simplified model, a set of quasiconvexity conditions at edges and corners of a (convex) polyhedral domain was employed to explain remarkable experimental observations in shape-memory alloys (see [BKS13]). We remark that quasiconvexity conditions at points of the boundary with conical singularities were first discussed in [BM84a, Remark 2] but, to the best of the authors' knowledge, this idea had not been previously applied (see Remark 4.3).

The plan of the paper is as follows: in Section 2, we state the general assumptions under which we establish our results. We further give all the required definitions on our domains as well as the generalised quasiconvexity conditions at the singular boundary. In Section 3, we show that all these quasiconvexity at the boundary conditions are indeed necessary for a strong local minimiser and, in Section 4, we state and prove the corresponding quasiconvexity-based sufficiency theorem which generalises the result of [GM09] to the case of domains locally diffeomorphic to a polytope. Our strategy, broadly motivated by the work in [CC], crucially uses a generalisation of Zhang's result on spatially-local minimisers. The proof of this generalisation, in the general case under consideration, becomes notationally involved and potentially difficult to follow its, otherwise, simple and clear ideas. Hence, for the convenience of the reader, the lemma is proved in the simpler case of the domain being itself a polytope, though the result is stated for the general case. The interested reader is referred to Section 5 for a full proof of the lemma.

2. PRELIMINARIES & DEFINITIONS

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain (i.e. open and connected) and consider an integral functional of the form

$$(2.1) \quad I(u) = \int_{\Omega} F(x, u(x), \nabla u(x)) dx,$$

which is to be minimised over a set of admissible maps \mathcal{A} , where $u: \Omega \rightarrow \mathbb{R}^N$ and the function $F: \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{N \times d} \rightarrow \mathbb{R}$ is continuous. The Weierstrass problem consists in finding necessary and sufficient conditions for a map u_0 to be a strong local minimiser of the functional I in \mathcal{A} , the space of admissible maps \mathcal{A} being part of the problem.

For our purposes, we define

$$(2.2) \quad \mathcal{A} := \{u \in C^1(\overline{\Omega}, \mathbb{R}^N) : u(x) = \bar{u}(x) \text{ for all } x \in \Gamma_D\},$$

where $\Gamma_D \subseteq \partial\Omega$ and \bar{u} is continuously differentiable on some open set in \mathbb{R}^d containing $\overline{\Gamma_D}$.

Note that if $u \in \mathcal{A}$, then $u(x) = \bar{u}(x)$ for all $x \in \overline{\Gamma_D}$. Hence, without loss of generality, we assume that Γ_D is the interior of $\overline{\Gamma_D}$, relative to $\partial\Omega$. By defining $\Gamma_N := \partial\Omega \setminus \overline{\Gamma_D}$, Γ_N is a relatively open subset of $\partial\Omega$ and $\partial\Omega = \Gamma_D \cup \overline{\Gamma_N}$. Indeed, if $x \in \partial\Omega \setminus \overline{\Gamma_N}$, then x has an open neighbourhood in $\partial\Omega$ that does not intersect Γ_N . Therefore, this neighbourhood must belong to the interior of $\overline{\Gamma_D}$, which is Γ_D . Additionally, we must require that Γ_D has itself a Lipschitz boundary in $\partial\Omega$ in the sense of [ADMD12, Definition 2.1]. This assumption allows us to make the identification (see [ADMD12, Proposition 6.2])

$$\begin{aligned} & \overline{\{u \in C^\infty(\overline{\Omega}, \mathbb{R}^N) : u(x) = 0 \text{ on } \Gamma_D\}^{W^{1,p}}} \\ &= \{u \in W^{1,p}(\Omega, \mathbb{R}^N) : Tu(x) = 0 \text{ } \mathcal{H}^{d-1}\text{-a.e. on } \Gamma_D\}, \end{aligned}$$

where T denotes the trace operator.

We note that the subscripts D and N in Γ_D and Γ_N stand for Dirichlet and Neumann and are meant to help the reader associate the two parts of the boundary of Ω as the prescribed (Γ_D) and free (Γ_N) boundary, respectively.

Definition 2.1. A map $u_0 \in \mathcal{A}$ is a strong local minimiser of I in \mathcal{A} if there exists an $\varepsilon > 0$ such that, whenever $u \in \mathcal{A}$ and $\|u - u_0\|_\infty < \varepsilon$, it holds that $I(u) \geq I(u_0)$.

Equivalently, because the uniform topology on the space of continuous functions is metrizable, the notion of strong local minimisers can be expressed in terms of sequences.

Definition 2.2. Let the space of variations $\text{Var}(\mathcal{A})$ be given by

$$(2.3) \quad \text{Var}(\mathcal{A}) = \{\varphi \in C^1(\overline{\Omega}, \mathbb{R}^N) : \varphi(x) = 0 \text{ for all } x \text{ in } \Gamma_D\}.$$

We say that a sequence $\{\varphi_j\} \subset \text{Var}(\mathcal{A})$ is a strong variation if $\varphi_j \rightarrow 0$, as $j \rightarrow \infty$, uniformly in $x \in \Omega$.

Then, a map $u_0 \in \mathcal{A}$ is a strong local minimiser of I in \mathcal{A} if, and only if, for every strong variation $\{\varphi_j\}$ there exists $J > 0$ such that

$$(2.4) \quad I(u_0 + \varphi_j) \geq I(u_0) \quad \text{for all } j \geq J.$$

More generally, given an open set $\omega \subseteq \mathbb{R}^d$ such that $\omega \cap \Omega \neq \emptyset$, we consider the following space of variations defined in ω .

$$\text{Var}(\omega, \mathbb{R}^N) := \{\varphi \in C^1(\overline{\omega}, \mathbb{R}^N) : \varphi(x) = 0 \text{ for all } x \text{ in } (\Gamma_D \cap \overline{\omega}) \cup (\partial\omega \cap \Omega)\}.$$

We note that $\text{Var}(\mathcal{A}) = \text{Var}(\Omega, \mathbb{R}^N)$.

Next, for $u_0 \in \mathcal{A}$, let

$$\mathcal{R}(u_0) = \{ (u_0(x), \nabla u_0(x)) : x \in \overline{\Omega} \}.$$

Then, in addition to the continuity of F , we assume that for some $p \in [2, \infty)$

[H0] the partial derivatives of first and second order in (y, z) of $F(x, y, z)$, denoted by F_y , F_z , F_{yy} , F_{zz} and F_{yz} , exist and are continuous on $\overline{\Omega} \times \mathcal{U}$ where \mathcal{U} is an open and bounded neighbourhood of \mathcal{R} in $\mathbb{R}^N \times \mathbb{R}^{N \times d}$;

[H1] (growth conditions) for all $x \in \overline{\Omega}$, $y \in \mathbb{R}^N$ and $z \in \mathbb{R}^{N \times d}$

$$(a) \quad |F(x, y, z)| \leq C(y)(1 + |z|^p)$$

$$(b) \quad |F_z(x, y, z)| \leq C(y)(1 + |z|^{p-1})$$

and

$$(c) \quad |F_y(x, y, z)| \leq C(y)(1 + |z|^p),$$

where $C(y) > 0$ is in $L_{\text{loc}}^\infty(\mathbb{R}^N)$ and depends on F .

Remark 2.3. Following [GM09], we remark that under the p -growth assumed in [H1] (a), the notion of strong or weak local minimisers for I in \mathcal{A} remains the same if we enlarge the space of admissible maps to

$$\mathcal{A}' := \{ u \in C(\overline{\Omega}, \mathbb{R}^N) \cap W^{1,p}(\Omega, \mathbb{R}^N) : u(x) = \bar{u}(x) \text{ for all } x \in \Gamma_D \}.$$

This is due to the fact that \mathcal{A} is dense in \mathcal{A}' under the topology generated by the norm $\|u\|_\infty + \|\nabla u\|_p$ and the functional I is finite and continuous on \mathcal{A}' under this topology. In particular, this implies that given $\varphi \in \text{Var}(\omega, \mathbb{R}^N)$, by extending φ to Ω so that it takes the value 0 in $\Omega \setminus \omega$, we may assume that φ is in the closure of $\text{Var}(\mathcal{A})$ with respect to the above norm. Then, if u_0 is a strong local minimiser of I in \mathcal{A} , the minimality condition

$$I(u_0 + \varphi) \geq I(u_0)$$

also holds for these maps provided that $\|\varphi\|_\infty$ is small enough. We will often make use of this remark without always appealing to the above argument.

We use the notation $a \cdot b$ to denote the usual inner product on \mathbb{R}^N or $\mathbb{R}^{N \times d}$ depending on whether a, b are vectors or matrices, the inner product on $\mathbb{R}^{N \times d}$ being defined by $a \cdot b = \text{tr}(ab^T)$. Also, by a single x argument in F or any of its derivatives, we mean that it is being evaluated at $(x, u_0(x), \nabla u_0(x))$.

Under [H0] and [H1] (a) it is well known that necessary conditions for a map $u_0 \in \mathcal{A}$ to be a strong local minimiser of I are the following:

(I) satisfaction of the weak Euler–Lagrange equations, i.e.

$$\int_{\Omega} [F_y(x) \cdot \varphi(x) + F_z(x) \cdot \nabla \varphi(x)] dx = 0,$$

for all $\varphi \in \text{Var}(\mathcal{A})$;

(II) non-negativity of the second variation, i.e.

$$\int_{\Omega} [F_{yy}(x) \varphi(x) \cdot \varphi(x) + 2F_{yz}(x) \varphi(x) \cdot \nabla \varphi(x) + F_{zz}(x) \nabla \varphi(x) \cdot \nabla \varphi(x)] dx \geq 0,$$

for all $\varphi \in \text{Var}(\mathcal{A})$;

(III) for all $x_0 \in \overline{\Omega}$, $F(x_0, u_0(x_0), \cdot)$ is quasiconvex in the interior, i.e.

$$\int_B [F(x_0, u_0(x_0), \nabla u_0(x_0) + \nabla \varphi(x)) - F(x_0)] dx \geq 0,$$

for all $\varphi \in C_c^1(\overline{B}, \mathbb{R}^N)$ where B denotes the unit ball in \mathbb{R}^d ;

(IV) for all $x_0 \in \Gamma_N$ in the neighbourhood of which $\partial\Omega$ is C^1 , $F(x_0, u_0(x_0), \cdot)$ is quasiconvex at the (smooth) boundary, i.e.

$$\int_{B_{d-1, x_0}} [F(x_0, u_0(x_0), \nabla u_0(x_0) + \nabla \varphi(x)) - F(x_0)] dx \geq 0$$

for all $\varphi \in V_{d-1, x_0}$, where

$$V_{d-1, x_0} = \{ \varphi \in C^1(\overline{B_{d-1, x_0}}, \mathbb{R}^N) : \varphi(x) = 0 \text{ for all } x \in \partial B \cap \overline{B_{d-1, x_0}} \},$$

$B_{d-1, x_0} = \{ x \in B : x \cdot n(x_0) < 0 \}$, and $n(x_0)$ is the outward unit normal to $\partial\Omega$ at x_0 .

As mentioned above, Grabovsky and Mengesha [GM09] showed that, under additional hypotheses on F and for Ω of class C^1 , a strengthened version of the above conditions is in fact sufficient for a map $u_0 \in \mathcal{A}$ to be a strong local minimiser of I . In this work, we establish a sufficiency theorem for the case in which the domain is locally diffeomorphic to a polytope and, in order to make this definition precise, we introduce certain terminology.

Definition 2.4.

- A closed convex polytope \mathcal{P}^c in \mathbb{R}^d is a bounded set defined as the intersection of q ($q \geq d+1$) open half-spaces; that is, there exists a set of unit vectors $m_i \in \mathbb{R}^d$ and a corresponding set of scalars $b_i \in \mathbb{R}$, $i = 1, \dots, q$, such that

$$\mathcal{P}^c = \bigcap_{i=1}^q \{ x \in \mathbb{R}^d : x \cdot m_i \leq b_i \}.$$

- A polytope \mathcal{P} in \mathbb{R}^d is an open and connected set with the property that there exists a finite collection of closed convex polytopes $\{\mathcal{P}_1^c, \dots, \mathcal{P}_n^c\}$ such that

$$\mathcal{P} = \text{int} \left(\bigcup_{i=1}^n \mathcal{P}_i^c \right),$$

where $\text{int}(A)$ denotes the interior of the set A .

Remark 2.5.

- The boundary of a polytope \mathcal{P} , $\partial\mathcal{P}$, consists of a finite number of $(d-k)$ -dimensional faces, $k = 1, \dots, d$, a terminology which will be abbreviated to simply “ $(d-k)$ -faces”. The $(d-1)$ -faces correspond to the smooth part of the boundary whereas, for all $k = 2, \dots, d$, the boundary becomes singular, e.g. for $k = d-1$ or $k = d$, the 1-dimensional faces or 0-dimensional faces correspond to edges and vertices respectively.
- From Definition 2.4 it follows that, given a polytope \mathcal{P} , there is a set of outward pointing unit vectors $m_i \in \mathbb{R}^d$ and scalars $b_i \in \mathbb{R}$, $i = 1, \dots, q$ defining it. Moreover, each set of the form

$$\{ x \in \overline{\mathcal{P}} : x \cdot m_i = b_i \}$$

itself defines a $(d-1)$ -face and

$$\partial \mathcal{P} = \bigcup_{i=1}^q \{x \in \overline{\mathcal{P}} : x \cdot m_i = b_i\}.$$

Any $(d-2)$ -face is the intersection of precisely two $(d-1)$ -faces whereas, for $k = 3, \dots, d$, the $(d-k)$ -faces are the intersection of at least k $(d-1)$ -faces (see [Grü03]).

- We also remark that, since polytopes are not in general convex, the $(d-k)$ -faces for $k = 2, \dots, d$ can be either outwards pointing or inwards pointing. Using a convexity argument, this distinction can be made as soon as one is given a point x_0 on the $(d-k)$ -face in question as well as the outwards pointing normals m_i to the $(d-1)$ -faces intersecting on the given $(d-k)$ -face. Then, considering a neighbourhood V of x_0 that does not intersect any other $(d-k)$ face, and a point $x \in V$ given as the convex combination of any two points on the corresponding $(d-1)$ -faces, we make the convention that

the $(d-k)$ -face is outwards pointing if $x \cdot m_i \leq 0 \forall i$ and

the $(d-k)$ -face is inwards pointing if $x \cdot m_i \geq 0 \forall i$.

- Lastly, note that given a polytope \mathcal{P} , for any $k = 2, \dots, d$ there is no well-defined normal for a point on a $(d-k)$ -face. However, one may define a natural *set* of normals associated to any such point given by all normals to the $(d-1)$ -faces intersecting along the particular $(d-k)$ -face. Of course, every other point on the given $(d-k)$ -face has the same associated set of normals and hence these characterise the specific $(d-k)$ -face itself.

We are now in a position to make the definition of our domains precise:

Definition 2.6. *We say that an open and bounded set $\Omega \subset \mathbb{R}^d$ is locally diffeomorphic to a polytope if for every point $x_0 \in \partial\Omega$ there exists $r_{x_0} > 0$, a ball $B(x_0, r_{x_0})$ centred at x_0 of radius r_{x_0} , a polytope \mathcal{P}_{x_0} , a subset $\mathcal{D}_{x_0} \subseteq \overline{\mathcal{P}_{x_0}}$ and a diffeomorphism $g_{x_0} : B(x_0, r_{x_0}) \cap \overline{\Omega} \rightarrow \mathcal{D}_{x_0}$, with the following properties:*

- g_{x_0} preserves orientation, i.e. $\det \nabla g_{x_0}(x) > 0$ for all $x \in B(x_0, r_{x_0}) \cap \overline{\Omega}$;
- $g_{x_0}(B(x_0, r_{x_0}) \cap \Omega) \subset \mathcal{P}_{x_0}$, $g_{x_0}(B(x_0, r_{x_0}) \cap \partial\Omega) \subset \partial \mathcal{P}_{x_0}$ and
- $g_{x_0}(B(x_0, r_{x_0}) \cap \Omega)$ intersects one, and only one, lowest dimensional boundary of \mathcal{P}_{x_0} , i.e. if the face of lowest dimension intersected by $g_{x_0}(B(x_0, r_{x_0}) \cap \overline{\Omega})$ is $(d-k_0)$, then there is a unique $(d-k_0)$ -face with such a property.

Throughout the remainder of this paper, $\Omega \subset \mathbb{R}^d$ is assumed to be locally diffeomorphic to a polytope and we agree to use the following terminology:

Definition 2.7. *We say that a point $x_0 \in \partial\Omega$ is $(d-k)$ -facelike for some $k \in \{1, \dots, d\}$ if $g_{x_0}(x_0)$ belongs to a $(d-k)$ -face of the polytope \mathcal{P}_{x_0} and $d-k$ is the minimum dimension of the faces in which $g_{x_0}(x_0)$ is found.*

Suppose that $x_0 \in \partial\Omega$ is $(d-1)$ -facelike. Then, in a neighbourhood of x_0 , $\partial\Omega$ must be of class C^1 and there exists a well-defined outward unit normal $n(x_0)$. Note that the map g_{x_0} maps x_0 to a $(d-1)$ -face of \mathcal{P}_{x_0} , say with outward unit normal to $g_{x_0}(x_0)$ given by $m(g_{x_0}(x_0)) \in S^{d-1}$. Then the two normals are related by

$$m(g_{x_0}(x_0)) // (\text{cof } \nabla g_{x_0}(x_0)) n(x_0) // (\nabla g_{x_0}(x_0))^{-T} n(x_0),$$

where $\text{cof} A$ is the $d \times d$ matrix of the $(d-1) \times (d-1)$ minors of A and we have used the standard identity $\text{cof} A = (\det A) A^{-T}$. Hence,

$$n(x_0) // (\nabla g_{x_0}(x_0))^T m(g_{x_0}(x_0)).$$

Similarly, let $x_0 \in \partial\Omega$ be a $(d-2)$ -facelike point. There is no well-defined normal at x_0 but $g_{x_0}(x_0)$ lies on a $(d-2)$ -face of \mathcal{P}_{x_0} and, as discussed above, this $(d-2)$ -face has a natural set of normals associated with it, these being given by the outward unit normals of the two $(d-1)$ -faces intersecting along the $(d-2)$ -face of \mathcal{P}_{x_0} where $g_{x_0}(x_0)$ lies, say $m_i(g_{x_0}(x_0))$, $i = 1, 2$. By the above reasoning, there are two associated normals to $x_0 \in \partial\Omega$, say $n_i(x_0)$, $i = 1, 2$ given by

$$n_i(x_0) // (\nabla g_{x_0}(x_0))^T m_i(g_{x_0}(x_0)), \quad i = 1, 2.$$

In analogy, for $k = 3, \dots, d$, a $(d-k)$ -face is the intersection of $l \geq k$ $(d-1)$ -faces and hence, if $x_0 \in \partial\Omega$ is a $(d-k)$ -facelike point, there are l natural normals $n_i(x_0)$ associated to x_0 . These are given by

$$n_i(x_0) // (\nabla g_{x_0}(x_0))^T m_i(g_{x_0}(x_0)), \quad i = 1, \dots, l,$$

where $m_i(g_{x_0}(x_0))$, $i = 1, \dots, l$ are the outward unit normals to the l $(d-1)$ -faces of \mathcal{P}_{x_0} intersecting along the given $(d-k)$ -face of \mathcal{P}_{x_0} . These normals are well-defined and this leads to the following definition:

Definition 2.8. For $k \in \{2, \dots, d\}$, let $x_0 \in \partial\Omega$ be a $(d-k)$ -facelike point so that $g_{x_0}(x_0)$ belongs to an outwards or inwards pointing $(d-k)$ -face of \mathcal{P}_{x_0} which is itself the intersection of l $(d-1)$ -faces with associated outward unit normals

$$m_1(g_{x_0}(x_0)), \dots, m_l(g_{x_0}(x_0)) \in \mathbb{R}^d.$$

The set of outward unit normals associated to the point x_0 is defined as the set of unit vectors $n_i(x_0)$, $i = 1, \dots, l$ given by

$$n_i(x_0) = \frac{(\nabla g_{x_0}(x_0))^T m_i(g_{x_0}(x_0))}{|(\nabla g_{x_0}(x_0))^T m_i(g_{x_0}(x_0))|}.$$

Every point $x \in B(x_0, r_{x_0}) \cap \partial\Omega$ such that $g_{x_0}(x)$ lies on the same $(d-k)$ -face as $g_{x_0}(x_0)$ has the same associated normals and we call the set of all such points an outwards or inwards pointing $(d-k)$ -facelike boundary with associated normals $n_i(x_0)$, $i = 1, \dots, l$.

We may now define the quasiconvexity at the boundary conditions:

Definition 2.9. Let $F: \mathbb{R}^{N \times d} \rightarrow \mathbb{R}$, $A \in \mathbb{R}^{N \times d}$ and denote by B the unit ball in \mathbb{R}^d .

- We say that F is quasiconvex in the interior at A if

$$\int_B [F(A + \nabla \varphi(x)) - F(A)] \, dx \geq 0$$

for all $\varphi \in C_c^1(B, \mathbb{R}^N)$.

- We say that F is quasiconvex at A at a $(d-k)$ -facelike boundary with associated set of normals $\{n_i(x_0)\}_{i=1}^l$ if

$$\int_{B_{d-k, x_0}} [F(A + \nabla \varphi(x)) - F(A)] \, dx \geq 0$$

for all $\varphi \in V_{d-k, x_0}$, where x_0 is any point on the $(d-k)$ -facelike boundary,

$$V_{d-k, x_0} := \{ \varphi \in C^1(\overline{B_{d-k, x_0}}, \mathbb{R}^N) : \varphi(x) = 0 \text{ for all } x \in \partial B \cap \overline{B_{d-k, x_0}} \},$$

and

$$B_{d-k,x_0} := \{x \in B : x \cdot n_i(x_0) < 0, \text{ for all } i = 1, \dots, l\},$$

if the facelike boundary is outwards pointing, whereas

$$B_{d-k,x_0} := \{x \in B : x \cdot n_i(x_0) < 0, \text{ for some } i = 1, \dots, l\},$$

if the facelike boundary is inwards pointing.

Note that, for convenience, we made no notational distinction between outwards and inwards pointing facelike boundaries. As explained in Remark 2.5, the knowledge of a point lying on the $(d-k)$ -face along with the associated set of normals to that face is enough to determine whether the boundary is outwards or inwards pointing. Hence, this is implicit in its definition.

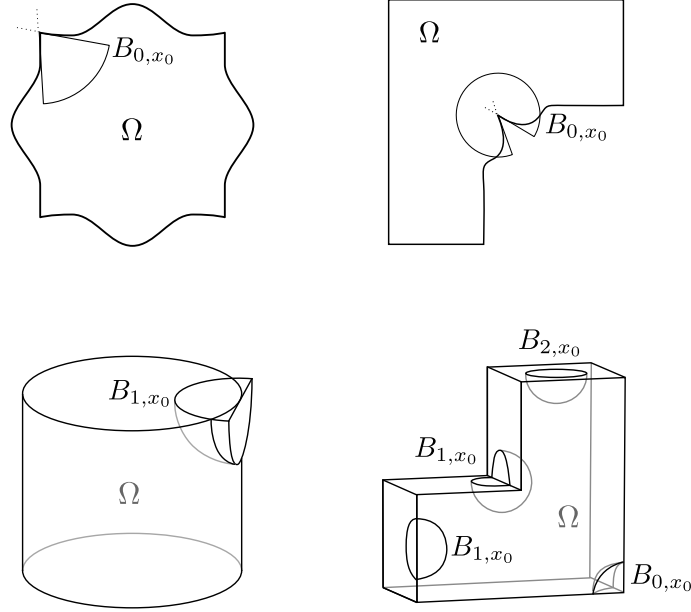


FIGURE 1. A visualisation of the sets B_{d-k,x_0} in the definition of quasiconvexity at the boundary, translated and scaled onto the respective parts of Ω . The corresponding space of test functions V_{d-k,x_0} simply amounts to smooth maps on the respective section of the unit ball which vanish on the part of the boundary that also belongs to the closed unit ball; that is, they vanish on the curved boundary whereas the (piecewise) planar part of the boundary is free.

Remark 2.10. We remark that by [H1] (a) F satisfies a p -growth. Then, the quasiconvexity conditions in the interior and at the $(d-k)$ -facelike boundaries in Definition 2.9 also hold for test functions $\varphi \in \overline{C_c^1(B, \mathbb{R}^N)}^{W^{1,p}} = \overline{W_0^{1,p}(B, \mathbb{R}^N)}$ and $\varphi \in \overline{V_{d-k,x_0}}^{W^{1,p}}$, respectively, where the closure is taken in the strong topology of $W^{1,p}$. The proof in the case of the interior can be found in [BM84b, Proposition 2.4] and the proof for the conditions at the boundary is essentially the same. This remark is crucial in the analysis that follows. As with Remark 2.3, if $\varphi \in C_c^1(\omega, \mathbb{R}^N)$ for some $\omega \subset B$, by extending φ to B such that it takes the value 0 outside $B \setminus \omega$, we obtain a map in $\overline{W_0^{1,p}(B, \mathbb{R}^N)}$ and the quasiconvexity condition still holds for these maps. The analogous situation holds for the case of quasiconvexity at the boundary and the space $\overline{V_{d-k,x_0}}^{W^{1,p}}$.

It is also well-known (see e.g. [Giu03, Remark 5.1]) that quasiconvexity in the interior is independent of the particular choice of the set on which integration is performed (in our case the unit

ball B). Indeed, if the condition holds for B , it also holds for any bounded, open set D . Similarly, quasiconvexity at a $(d-k)$ -facelike boundary also does not depend on the specific form of the sets B_{d-k,x_0} and we make this precise in the following definition and lemma, motivated by [BM84a].

Definition 2.11. *A standard outwards, respectively inwards, $(d-k)$ -boundary region with the set of normals $\{n_i\}_{i=1}^l$ is a bounded Lipschitz domain $D \subset \mathbb{R}^d$ with the following properties:*

- (i) *D is contained in the intersection, respectively in the union, of the l half-spaces*

$$K_{n_1, \dots, n_l}^a = \bigcap_{i=1}^l \left\{ x \in \mathbb{R}^d : x \cdot n_i < a_i \right\} \quad (\text{outwards})$$

$$K_{n_1, \dots, n_l}^a = \bigcup_{i=1}^l \left\{ x \in \mathbb{R}^d : x \cdot n_i < a_i \right\} \quad (\text{inwards})$$

where $a_i = a \cdot n_i$ for some $a \in \mathbb{R}^d$.

- (ii) *the $(d-k)$ -dimensional interior of*

$$\partial D \cap \left\{ x \in \mathbb{R}^d : x \cdot n_i = a_i \right\}$$

is non-empty for every $1 \leq i \leq l$. The remaining part of the boundary of D is then given by $\partial D \cap K_{n_1, \dots, n_l}^a$.

Lemma 2.12. *Let $F: \mathbb{R}^{N \times d} \rightarrow \mathbb{R}$ and $A \in \mathbb{R}^{N \times d}$. If F is quasiconvex at A at an outwards, respectively inwards, $(d-k)$ -facelike boundary with the normals $\{n_i\}_{i=1}^l$ in the sense of Definition 2.9 then, for any standard outwards, respectively inwards, $(d-k)$ -boundary region D with the set of normals $\{n_i\}_{i=1}^l$, it holds that*

$$\int_D [F(A + \nabla \varphi(x)) - F(A)] dx \geq 0,$$

for all $\varphi \in V_D$, where

$$V_D = \left\{ \varphi \in C^1(\overline{D}, \mathbb{R}^N) : \varphi(x) = 0 \text{ for all } x \in \partial D \cap K_{n_1, \dots, n_l}^a \right\}.$$

Proof. Suppose that F is quasiconvex at A at an outwards $(d-k)$ -facelike boundary with the normals $\{n_i\}_{i=1}^l$ as in Definition 2.9. The inwards case is completely analogous. Let D be a corresponding standard (outwards) $(d-k)$ -boundary region and choose d_0 in the non-empty $(d-k)$ -dimensional interior of the $(d-k)$ -dimensional face in $\partial D \cap \partial K_{n_1, \dots, n_l}^a$. Setting $\tilde{D} = D - d_0$, we note that $d_0 \cdot n_i = a_i$ for all $i = 1, \dots, l$ and hence \tilde{D} is another standard (outwards) $(d-k)$ -boundary region with the normals $\{n_i\}$ and $a = 0 \in \mathbb{R}^d$ in the definition of the relevant half-spaces.

Note that the point $0 \in \mathbb{R}^d$ is contained in $B \cap \partial B_{d-k, d_0}$ and hence, for $\varepsilon > 0$ small enough,

$$\begin{aligned} \varepsilon \tilde{D} &\subset B_{d-k, d_0}, \\ (\varepsilon \partial \tilde{D}) \cap K_{n_1, \dots, n_l}^0 &\subset B_{d-k, d_0}, \\ (\varepsilon \partial \tilde{D}) \cap \partial K_{n_1, \dots, n_l}^0 &\subset B \cap \partial B_{d-k, d_0}. \end{aligned}$$

Then, given $\varphi \in V_D$, define $\psi: B_{d-k, d_0} \rightarrow \mathbb{R}^N$ by

$$\psi(x) := \begin{cases} \varepsilon \varphi(d_0 + x/\varepsilon), & x \in \tilde{D} = \varepsilon(D - d_0) \subset B_{d-k, d_0} \\ 0, & \text{otherwise.} \end{cases}$$

Then $\psi \in \overline{V_{d-k,d_0}^{W^{1,p}}}$ and, by Remark 2.10, the quasiconvexity condition applies, i.e.

$$\begin{aligned} 0 &\leq \int_{B_{d-k,d_0}} [F(A + \nabla \psi(x)) - F(A)] dx = \int_{\bar{D}} [F(A + \nabla \phi(d_0 + x/\varepsilon)) - F(A)] dx \\ &= \varepsilon^d \int_D [F(A + \nabla \phi(y)) - F(A)] dx. \end{aligned}$$

□

3. NECESSARY CONDITIONS

Equipped with all the required definitions, we proceed to discuss the necessity of quasiconvexity at the $(d-k)$ -facelike boundary for a map $u_0 \in \mathcal{A}$ to be a strong local minimiser of the functional I . We recall that

$$I(u) = \int_{\Omega} F(x, u(x), \nabla u(x)) dx$$

and

$$\mathcal{A} = \{u \in C^1(\bar{\Omega}, \mathbb{R}^N) : u(x) = \bar{u}(x) \text{ for all } x \in \Gamma_D\}.$$

We begin by compiling all the necessary conditions in one theorem. We also recall that a single x argument in F , or any of its derivatives, corresponds to the triple $(x, u_0(x), \nabla u_0(x))$.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^d$ be locally diffeomorphic to a polytope. For $u_0 \in \mathcal{A}$, assume that $F: \bar{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{N \times d} \rightarrow \mathbb{R}$ is continuous and satisfies [H0] and [H1]. If u_0 is a strong local minimiser of I in \mathcal{A} , then the following hold:*

(I) u_0 satisfies the weak Euler–Lagrange equations, i.e.

$$\int_{\Omega} [F_y(x) \cdot \phi(x) + F_z(x) \cdot \nabla \phi(x)] dx = 0,$$

for all $\phi \in \text{Var}(\mathcal{A})$;

(II) the second variation at u_0 is non-negative, i.e.

$$\int_{\Omega} [F_{yy}(x) \phi(x) \cdot \phi(x) + 2F_{yz}(x) \nabla \phi(x) \cdot \phi(x) + F_{zz}(x) \nabla \phi(x) \cdot \nabla \phi(x)] dx \geq 0,$$

for all $\phi \in \text{Var}(\mathcal{A})$;

(III) for all $x_0 \in \bar{\Omega}$, $F(x_0, u_0(x_0), \cdot)$ is quasiconvex in the interior, i.e.

$$\int_B [F(x_0, u_0(x_0), \nabla u_0(x_0) + \nabla \phi(x)) - F(x_0)] dx \geq 0,$$

for all $\phi \in C_c^1(B, \mathbb{R}^N)$;

(IV)_k for all $k = 1, \dots, d$ and all $(d-k)$ -facelike points $x_0 \in \Gamma_N$, $F(x_0, u_0(x_0), \cdot)$ is quasiconvex at the $(d-k)$ -facelike boundary, i.e.

$$\int_{B_{d-k,x_0}} [F(x_0, u_0(x_0), \nabla u_0(x_0) + \nabla \phi(x)) - F(x_0)] dx \geq 0,$$

for all $\phi \in V_{d-k,x_0}$.

Proof of Theorem 3.1. The proofs of (I), (II) are standard and can be found in e.g. [Dac08]. The necessity of (III) is essentially due to Meyers [Mey65] (see also [BM84a]). The proof of (IV)₁, amounting to the standard quasiconvexity at the boundary condition, can be found in Ball & Marsden [BM84a]. Nevertheless, here, we give an alternative proof which is also appropriate for proving the quasiconvexity at the singular points of the boundary. To this end, let $x_0 \in \Gamma_N$ be a $(d-k)$ -facelike point with associated set of outward unit normals $\{n_i(x_0)\}_{i=1}^l$. Note that if $k = 1$

or $k = 2$ then $l = 1$ and $l = 2$ respectively. Let g_{x_0} be the respective diffeomorphism and, to ease notation, write $g_{x_0} = g$ and $A := \nabla g(x_0)$, noting that $\det A > 0$. Also, let $\varphi \in V_{d-k, x_0}$ and define

$$u_\varepsilon(x) = \begin{cases} u_0(x) + \varepsilon \varphi \left(A^{-1} \frac{g(x) - g(x_0)}{\varepsilon} \right), & x \in g^{-1}(g(x_0) + \varepsilon AB_{d-k, x_0}) \\ u_0(x), & \text{otherwise in } \Omega. \end{cases}$$

We remark that for $\varepsilon > 0$ small enough

$$\begin{aligned} g^{-1}(g(x_0) + \varepsilon AB_{d-k, x_0}) &\subset \Omega, \\ g^{-1}(g(x_0) + \varepsilon(A\partial B \cap \overline{B_{d-k, x_0}})) &\subset \Omega \text{ and} \\ g(x_0) + \varepsilon(AB \cap \overline{B_{d-k, x_0}}) &\subset \bigcap_{i=1}^l \left\{ p \in \partial \mathcal{P} : q \cdot \text{cof} A n_i(x_0) = g(x_0) \cdot \text{cof} A n_i(x_0) \right\}. \end{aligned}$$

The two first inclusions are trivial to see and, regarding the third one, note that if $b \in B \cap \overline{B_{d-k, x_0}}$ then $b \cdot n_i(x_0) = 0$ for all $i = 1, \dots, l$. Hence, for such b and all i ,

$$\begin{aligned} (g(x_0) + \varepsilon Ab) \cdot \text{cof} A n_i(x_0) &= g(x_0) \cdot \text{cof} A n_i(x_0) + \varepsilon Ab \cdot \text{cof} A n_i(x_0) \\ &= g(x_0) \cdot \text{cof} A n_i(x_0) + \varepsilon \det A b \cdot n_i(x_0) \\ &= g(x_0) \cdot \text{cof} A n_i(x_0). \end{aligned}$$

In particular, this says that $g(x_0) + \varepsilon(AB \cap \overline{B_{d-k, x_0}})$ - the (piecewise) flat part of the boundary - lies in fact on the $(d-k)$ -face of $\mathcal{P} = \mathcal{P}_{x_0}$ where $g(x_0)$ is, i.e. the boundary of $g(x_0) + \varepsilon AB_{d-k, x_0}$ has two parts, one lying in the interior of \mathcal{P} and given by $g(x_0) + \varepsilon(A\partial B \cap \overline{B_{d-k, x_0}})$ and one lying on the $(d-k)$ -face in question given by $g(x_0) + \varepsilon(AB \cap \overline{B_{d-k, x_0}})$. Under the diffeomorphism g^{-1} these sets of course get mapped to the interior of Ω and to Γ_N respectively.

Due to the above reasoning, u_ε lies in the closure of \mathcal{A} in the topology generated by the norm $\|u\|_\infty + \|\nabla u\|_p$ and $\|u_\varepsilon - u_0\|_\infty = \varepsilon \|\varphi\|_\infty \rightarrow 0$, as $\varepsilon \rightarrow 0$. Hence, by Remark 2.3 and the fact that u_0 is a strong local minimiser, we infer that

$$\begin{aligned} 0 &\leq I(u_\varepsilon) - I(u_0) \\ &= \int_{g^{-1}(g(x_0) + \varepsilon AB_{d-k, x_0})} (F(x, u_\varepsilon(x), \nabla u_\varepsilon(x)) - F(x)) \, dx. \end{aligned}$$

Note that

$$\nabla u_\varepsilon(x) = \nabla u_0(x) + \nabla \varphi(A^{-1}(g(x) - g(x_0))/\varepsilon)A^{-1}\nabla g(x)$$

and change variables to

$$y_\varepsilon(x) = A^{-1}(g(x) - g(x_0))/\varepsilon$$

with inverse $x_\varepsilon(y) = g^{-1}(g(x_0) + \varepsilon Ay)$ and $\nabla x_\varepsilon(y) = \varepsilon \nabla g^{-1}(g(x_0) + \varepsilon Ay)A$ to get that

$$\begin{aligned} 0 &\leq \varepsilon^d \int_{B_{d-k, x_0}} [F(x_\varepsilon(y), u_0(x_\varepsilon(y)) + \varepsilon \varphi(y), \nabla u_0(x_\varepsilon(y)) + \nabla \varphi(y)A^{-1}\nabla g(x_\varepsilon(y))) \\ &\quad - F(x_\varepsilon(y), u_0(x_\varepsilon(y)), \nabla u_0(x_\varepsilon(y)))] \det[\nabla g^{-1}(g(x_0) + \varepsilon Ay)A] \, dx. \end{aligned}$$

Dividing by ε^d and sending $\varepsilon \rightarrow 0$, bounded convergence gives that

$$\begin{aligned} 0 &\leq \int_{B_{d-k, x_0}} [F(x_0, u_0(x_0), \nabla u_0(x_0) + \nabla \varphi(y)) - F(x_0)] \det[(\nabla g(x_0))^{-1}A] \, dx \\ &= \int_{B_{d-k, x_0}} [F(x_0, u_0(x_0), \nabla u_0(x_0) + \nabla \varphi(y)) - F(x_0)] \, dx, \end{aligned}$$

noting that $\nabla g^{-1}(g(x_0)) = (\nabla g(x_0))^{-1}$ and recalling that $A = \nabla g(x_0)$. This proves $(IV)_k$. \square

4. SUFFICIENT CONDITIONS

In this section we prove our main result in the form of a quasiconvexity-based sufficiency theorem for a map $u_0 \in \mathcal{A}$ to be a strong local minimiser of I in \mathcal{A} . For this we require additional coercivity conditions on the integrand F which we state here. We recall that $p \in [2, \infty)$.

[H2] (coercivity conditions) F is bounded from below. If $p = 2$, we assume that

$$\int_{\Omega} F(x, u(x), \nabla u(x)) \, dx \geq c_2(r) \|u\|_{1,p}^p - c_1(r)$$

for all $u \in \mathcal{A}$ such that $\|u\|_{\infty} \leq r$, where $c_1(r) > 0$ and $c_2(r) > 0$ are locally bounded. If $p > 2$, we assume that for all $\varphi \in \text{Var}(\mathcal{A})$ with $\|\varphi\|_{\infty} \leq r$,

$$\int_{\Omega} [F(x, u_0(x) + \varphi(x), \nabla u_0(x) + \nabla \varphi(x)) - F(x)] \, dx \geq c_1(r) \|\nabla \varphi\|_p^p - c_2(r) \|\nabla \varphi\|_2^2$$

for some $c_1(r) > 0$, $c_2(r) > 0$ which are locally bounded. Note that the latter condition need only apply to u_0 .

Lastly, we require an additional uniform continuity assumption:

[UC] For every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every $z \in \mathbb{R}^{N \times d}$ and $x, x_0 \in \overline{\Omega}$ with $|x - x_0| < \delta$, it holds that

$$|F(x_0, u_0(x_0), \nabla u_0(x_0) + z) - F(x, u_0(x), \nabla u_0(x) + z)| < \varepsilon(1 + |z|^p).$$

For brevity we also introduce the function S below.

Definition 4.1. For $k \in \mathbb{N}$, let $S: \mathbb{R}^k \rightarrow \mathbb{R}$ denote the function

$$S(\xi) = (|\xi|^2 + |\xi|^p)^{\frac{1}{2}}.$$

We are now in a position to state our main theorem:

Theorem 4.2 (Sufficiency Theorem). Let $\Omega \subset \mathbb{R}^d$ be locally diffeomorphic to a polytope. Let $u_0 \in \mathcal{A}$ and assume that $F: \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{N \times d} \rightarrow \mathbb{R}$ is continuous and satisfies [H0]–[H2]. Further assume that for some $c_0 > 0$ the following hold:

(I) u_0 satisfies the weak Euler–Lagrange equations, i.e., for all $\varphi \in \text{Var}(\mathcal{A})$,

$$\int_{\Omega} [F_y(x) \cdot \varphi(x) + F_z(x) \cdot \nabla \varphi(x)] \, dx = 0;$$

(II) the second variation at u_0 is strongly positive, i.e., for all $\varphi \in \text{Var}(\mathcal{A})$,

$$\int_{\Omega} [F_{yy}(x) \varphi(x) \cdot \varphi(x) + 2F_{yz}(x) \nabla \varphi(x) \cdot \varphi(x) + F_{zz}(x) \nabla \varphi(x) \cdot \nabla \varphi(x)] \, dx \geq c_0 \|\nabla \varphi\|_2^2;$$

(III) for all $x_0 \in \overline{\Omega}$, $F(x_0, u_0(x_0), \cdot)$ is strongly quasiconvex in the interior, i.e., for all $\varphi \in C_c^1(B, \mathbb{R}^N)$,

$$\int_B [F(x_0, u_0(x_0), \nabla u_0(x_0) + \nabla \varphi(x)) - F(x_0)] \, dx \geq c_0 \|S(\nabla \varphi)\|_2^2;$$

(IV)_k for all $(d - k)$ -facelike points $x_0 \in \Gamma_N$, $F(x_0, u_0(x_0), \cdot)$ is strongly quasiconvex at the $(d - k)$ -facelike boundary, i.e., for all $\varphi \in V_{d-k, x_0}$,

$$\int_{B_{d-k, x_0}} [F(x_0, u_0(x_0), \nabla u_0(x_0) + \nabla \varphi(x)) - F(x_0)] \, dx \geq c_0 \|S(\nabla \varphi)\|_2^2;$$

(V) u_0 satisfies [UC].

Then, u_0 is a strong local minimiser of I in \mathcal{A} .

Remark 4.3. Theorem 4.2 and the proof that we present here remain valid if we allow conical singularities on the free boundary of the domain. The analogous quasiconvexity at the boundary condition can be defined and shown to be necessary following the same strategy as in Theorem 4.2. There is no finite set of normals defining the domain of integration for quasiconvexity in this case. However, if x_0 is the vertex of a cone C , we integrate over $C \cap B(x_0, 1)$ and we consider variations that are equal to 0 on $\partial B(x_0, 1) \cap C$ to obtain the corresponding necessary quasiconvexity condition. Furthermore, these arguments can be extended to consider domains that are locally diffeomorphic to finite unions of cones and polytopes.

The proof of the Sufficiency Theorem is based on Theorem 4.4 below. In its original form due to Zhang [Zha92], the result asserts that, under the quasiconvexity assumptions, any C^1 solution of the Euler–Lagrange equations is a spatially-local minimiser. In the presence of lower order terms in F , we use a variant appropriate for our purposes.

Theorem 4.4. *Let $\Omega \subset \mathbb{R}^d$ be locally diffeomorphic to a polytope. Let $u_0 \in C^1(\overline{\Omega}, \mathbb{R}^N)$ and assume that $F: \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{N \times d} \rightarrow \mathbb{R}$ is continuous and satisfies [H0]–[H2]. In the notation of Theorem 4.2, suppose that conditions (I), (III) and $(IV)_k$ for all $k = 1, \dots, d$ are satisfied. Then, there exist some $R > 0$ and $\delta > 0$ such that, denoting by $\Omega(x_0, R) := \Omega \cap B(x_0, R)$, the following hold:*

- (1) *for all $x_0 \in \overline{\Omega}$*
- $$(*) \quad \begin{aligned} & \frac{c_0}{2} \int_{\Omega(x_0, R)} (|S(\nabla \varphi(x))|^2 - C|\varphi(x)|^2) \, dx \\ & \leq \int_{\Omega(x_0, R)} (F(x, u_0(x) + \varphi(x), \nabla u_0(x) + \nabla \varphi(x)) - F(x)) \, dx \end{aligned}$$
- for all $\varphi \in W_0^{1,p}(\Omega(x_0, R), \mathbb{R}^N)$ with $\|\varphi\|_{L^\infty} < \delta$;*
- (2) *for all $x_0 \in \overline{\Omega}$ such that $\Omega(x_0, R) \cap \Gamma_N \neq \emptyset$, if $d - k$ is the minimum dimension such that $\Omega(x_0, R)$ intersects a $(d - k)$ -facelike boundary and $\Omega(x_0, R)$ intersects precisely one $(d - k)$ -facelike boundary, then $(*)$ holds for every function $\varphi \in \overline{\text{Var}(\Omega(x_0, R), \mathbb{R}^N)^{W^{1,p}}}$ satisfying $\|\varphi\|_{L^\infty} < \delta$, where the closure is taken in $W^{1,p}(\Omega, \mathbb{R}^N)$ with the strong topology.*

Before proceeding with the proof of this theorem, we prove the following technical Lemma regarding the growth conditions that the linearisation of F satisfies.

Lemma 4.5. *Let $\Omega \subset \mathbb{R}^d$ be locally diffeomorphic to a polytope. Assume further that $F: \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{N \times d} \rightarrow \mathbb{R}$ is a continuous integrand, $u_0 \in C^1(\overline{\Omega}, \mathbb{R}^N)$ and that [H0], [H1] and [UC] hold. Define the function $G: \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{N \times d} \rightarrow \mathbb{R}$ by*

$$\begin{aligned} G(x, y, z) &:= F(x, u_0(x) + y, \nabla u_0(x) + z) - F(x) - F_y(x) \cdot y - F_z(x) \cdot z \\ &= \int_0^1 (1-t) L(x, u_0(x) + ty, \nabla u_0(x) + tz) [(y, z), (y, z)], \end{aligned}$$

where the bilinear form $L(x, v, w)$ is given by

$$\begin{aligned} L(x, v, w) [(y, z), (\hat{y}, \hat{z})] &:= F_{yy}(x, v, w) y \cdot \hat{y} + F_{yz}(x, v, w) y \cdot \hat{z} + F_{yz}(x, v, w) \hat{y} \cdot z \\ &\quad + F_{zz}(x, v, w) z \cdot \hat{z}. \end{aligned}$$

For each $x \in \overline{\Omega}$, $y, \hat{y} \in \mathbb{R}^N$ and $z, \hat{z} \in \mathbb{R}^{N \times d}$, the function G satisfies

- (a) $|G(x, y, z)| \leq C(y) (|S(y)|^2 + |S(z)|^2)$ for some locally bounded function $C(y)$.

(b) For every $\varepsilon > 0$ there exist $R = R_\varepsilon > 0$ and $c = c_\varepsilon > 0$ such that, if $|x - x_0| < R$, then

$$|G(x, y, z) - G(x_0, 0, z)| \leq c_\varepsilon C(y) (|y|^2 + |S(z)|^2 |y|) + (c\omega(|y|) + \varepsilon) |S(z)|^2,$$

where ω is an increasing, continuous function with $\omega(0) = 0$ and $C(y)$ is locally bounded.

(c) $|G(x, y, z) - G(x, \hat{y}, \hat{z})| \leq C(y, \hat{y}) (A_{p-1}(y, z, \hat{y}, \hat{z})|z - \hat{z}| + A_p(y, z, \hat{y}, \hat{z})|y - \hat{y}|)$ for some locally bounded function $C(y, \hat{y})$ on \mathbb{R}^{2N} and

$$A_p(y, z, \hat{y}, \hat{z}) = |y| + |\hat{y}| + |z| + |\hat{z}| + |z|^p + |\hat{z}|^p.$$

Proof. The proof of this lemma is motivated by the truncation strategy originated in [AF87].

To prove (a), first consider the case $|y| + |z| \leq 1$. By the continuity of the second partial derivatives of F we obtain that

$$|G(x, y, z)| \leq \int_0^1 |L(x, u_0(x) + ty, \nabla u_0(x) + tz)[(y, z), (y, z)]| \leq c(|y|^2 + |z|^2)$$

for some $c = c(d, N, \|u_0\|_{C^1})$.

On the other hand, if $|y| + |z| > 1$, the triangle inequality together with [H0], [H1] imply that

$$\begin{aligned} |G(x, y, z)| &\leq |F(x, u_0(x) + y, \nabla u_0(x) + z) - F(x, u_0(x), \nabla u_0(x) + z)| \\ &\quad + |F(x, u_0(x), \nabla u_0(x) + z) - F(x, u_0(x), \nabla u_0(x))| \\ &\quad + |F_y(x, u_0(x), \nabla u_0(x))||y| + |F_z(x, u_0(x), \nabla u_0(x))||z| \\ &\leq c(C(y)(1 + |z|^p) + |y| + (1 + |z|^{p-1})|z| + |z|) \\ &\leq c(C(y)(|y|^p + |z|^p) + (|y|^p + |z|^p)). \end{aligned}$$

The last estimate above follows from the fact that $|y|^{p-1} + |z|^{p-1} > c_p$ for some constant $c_p > 0$ and from the inequality $ab^{p-1} \leq a^p + b^p$ for $a, b > 0$. Using the definition of S , this proves (a).

For the proof of (b), we use the triangle inequality to obtain

$$|G(x, y, z) - G(x_0, 0, z)| \leq |G(x, y, z) - G(x, 0, z)| + |G(x, 0, z) - G(x_0, 0, z)| =: I + II.$$

We first analyse the term I by considering the following two cases:

Case 1. If $|y| + |z| \leq 1$, by assumption [H0] we infer that

$$\begin{aligned} I &\leq \int_0^1 |(L(x, u_0(x) + ty, \nabla u_0(x) + tz) - L(x, u_0(x), \nabla u_0(x) + tz))[(y, z), (y, z)]| \, dt \\ &\quad + \int_0^1 |L(x, u_0(x), \nabla u_0(x) + tz)[(y, 2z), (y, 0)]| \, dt \\ &\leq c\omega_0(|y|)(|y|^2 + |z|^2) + c(|y| + |z|)|y| \\ &\leq c\omega_0(|y|)(|y|^2 + |z|^2) + (c + c_\varepsilon)|y|^2 + \varepsilon|z|^2, \end{aligned}$$

where ω_0 is a modulus of continuity for the second order derivatives of F . We remark that the last estimate above follows for an arbitrary $\varepsilon > 0$ after applying Young's inequality.

Case 2. If $|y| + |z| > 1$, assumption [H1] implies that

$$\begin{aligned} I &= |F(x, u_0(x) + y, \nabla u_0(x) + z) - F(x, u_0(x), \nabla u_0(x) + z) - F_y(x, u_0(x), \nabla u_0(x)) \cdot y| \\ &= \int_0^1 [F_y(x, u_0(x) + ty, \nabla u_0(x) + z) - F_y(x, u_0(x), \nabla u_0(x))] \cdot y \, dt \\ &\leq C(y)(1 + |z|^p)|y| \leq C(y)(|y|^2 + 2|z|^2|y| + |z|^p|y|). \end{aligned}$$

To obtain the last inequality we consider the cases $|z| \geq \frac{1}{2}$ and $|z| < \frac{1}{2}$, which in turn implies $|y| \geq \frac{1}{2}$.

For the term II , in a similar spirit we have that if $|z| \leq 1$, then

$$\begin{aligned} II &\leq \int_0^1 |F_{zz}(x, u_0(x), \nabla u_0(x) + tz) - F_{zz}(x, u_0(x_0), \nabla u_0(x_0) + tz)| |z|^2 dt \\ &\leq c\tilde{\omega}_0(|x - x_0|)|z|^2, \end{aligned}$$

where $\tilde{\omega}_0$ is a modulus of continuity depending, in this case, also on the C^1 function u_0 .

If, on the other hand, $|z| > 1$, we observe that

$$\begin{aligned} II &\leq |F(x, u_0(x), \nabla u_0(x) + z) - F(x_0, u_0(x_0), \nabla u_0(x_0) + z)| \\ &\quad + |F(x_0, u_0(x_0), \nabla u_0(x_0)) - F(x, u_0(x), \nabla u_0(x))| \\ &\quad + |F_z(x, u_0(x), \nabla u_0(x)) - F_z(x_0, u_0(x_0), \nabla u_0(x_0))| |z|. \end{aligned}$$

From the uniform continuity assumption [UC] and the continuity of F_z and ∇u_0 , it is clear that, given $\varepsilon > 0$, there exists an $R = R(\varepsilon) > 0$ such that if $|x - x_0| < R$, then

$$(4.1) \quad II \leq \varepsilon(1 + |z|^p) + \varepsilon|z| \leq 2\varepsilon|z|^p$$

for $|z| > 1$. Bringing all the above estimates together we find that, for a given $\varepsilon > 0$, there exist an $R = R_\varepsilon > 0$ and a constant $c_\varepsilon > 0$ depending on ε and $\|u_0\|_{C^1}$, such that, if $|x - x_0| < R$, then

$$(4.2) \quad |G(x, y, z) - G(x_0, 0, z)| \leq c_\varepsilon C(y) (|y|^2 + |z|^2|y| + |z|^p|y|) + (c\omega(|y|) + \varepsilon) (|z|^2 + |z|^p),$$

where ω is an increasing, continuous function with $\omega(0) = 0$ and $C(y)$ is locally bounded. This concludes the proof of (b).

The proof of (c) follows in a similar fashion. We first use the triangle inequality to estimate

$$|G(x, y, z) - G(x, \hat{y}, \hat{z})| \leq |G(x, y, z) - G(x, \hat{y}, z)| + |G(x, \hat{y}, z) - G(x, \hat{y}, \hat{z})| =: I + II.$$

As before, we use a truncation strategy and consider two cases.

Case I: $|y| + |\hat{y}| + |z| + |\hat{z}| \leq 1$. The approach that we follow for this case is also motivated by the quadratic behaviour of G for small values of $|y| + |\hat{y}| + |z| + |\hat{z}|$. However, here we do not evoke the quadratic representation given by Taylor's Theorem, but the one that we obtain by applying twice the Fundamental Theorem of Calculus. The reason for this is that, in this way, we obtain sharper estimates that are explicitly used in the sufficiency theorem. The proof is performed as follows:

$$\begin{aligned} I &= \left| \int_0^1 (F_y(x, u_0(x) + \hat{y} + t(y - \hat{y}), \nabla u_0(x) + z) - F_y(x, u_0(x), \nabla u_0(x))) \cdot (y - \hat{y}) dt \right| \\ &\leq \left| \int_0^1 (F_y(x, u_0(x) + \hat{y} + t(y - \hat{y}), \nabla u_0(x) + z) - F_y(x, u_0(x), \nabla u_0(x) + z)) \cdot (y - \hat{y}) dt \right| \\ &\quad + \left| \int_0^1 (F_y(x, u_0(x), \nabla u_0(x) + z) - F_y(x, u_0(x), \nabla u_0(x))) \cdot (y - \hat{y}) dt \right| \\ &\leq \int_0^1 \int_0^1 |F_{yy}(x, u_0(x) + st(y - \hat{y}), \nabla u_0(x) + z)(t(y - \hat{y})) \cdot (y - \hat{y})| ds dt \\ &\quad + \int_0^1 \int_0^1 |F_{zy}(x, u_0(x), \nabla u_0(x) + sz)z \cdot (y - \hat{y})| ds dt \\ &\leq c(|y| + |\hat{y}| + |z| + |\hat{z}|)|y - \hat{y}|. \end{aligned}$$

Moreover,

$$\begin{aligned}
II &= \left| \int_0^1 (F_z(x, u_0(x) + \hat{y}, \nabla u_0(x) + \hat{z} + t(z - \hat{z})) - F_z(x, u_0(x), \nabla u_0(x))) \cdot (z - \hat{z}) dt \right| \\
&\leq \left| \int_0^1 (F_z(x, u_0(x) + \hat{y}, \nabla u_0(x) + \hat{z} + t(z - \hat{z})) - F_z(x, u_0(x), \nabla u_0(x) + \hat{z})) \cdot (z - \hat{z}) dt \right| \\
&\quad + \left| \int_0^1 (F_z(x, u_0(x), \nabla u_0(x) + \hat{z}) - F_z(x, u_0(x), \nabla u_0(x))) \cdot (z - \hat{z}) dt \right| \\
&\leq \int_0^1 \int_0^1 |F_{yz}(x, u_0(x) + s\hat{y}, \nabla u_0(x) + \hat{z} + t(z - \hat{z}))\hat{y} \cdot (z - \hat{z})| ds dt \\
&\quad + \int_0^1 \int_0^1 |F_{zz}(x, u_0(x), \nabla u_0(x) + s\hat{z})\hat{z} \cdot (z - \hat{z})| ds dt \\
&\leq c(|y| + |\hat{y}| + |z| + |\hat{z}|)|z - \hat{z}|.
\end{aligned}$$

We emphasize that the last inequality in each of the two estimates above relies only on condition [H0] and the fact that u_0 and ∇u_0 are uniformly bounded.

Case 2: $|y| + |\hat{y}| + |z| + |\hat{z}| > 1$. By the triangle inequality and the Fundamental Theorem of Calculus we obtain that

$$\begin{aligned}
I &\leq \int_0^1 |F_y(x, u_0(x) + \hat{y} + t(y - \hat{y}), \nabla u_0(x) + z) \cdot (y - \hat{y})| dt + |F_y(x) \cdot (y - \hat{y})| \\
&\leq C(\hat{y} + t(y - \hat{y}))(1 + |z|^p)|y - \hat{y}| + c|y - \hat{y}| \\
&\leq C(y, \hat{y})(|y| + |\hat{y}| + |z| + |\hat{z}| + |z|^p)|y - \hat{y}|.
\end{aligned}$$

The second inequality above follows from the growth condition [H1] (b).

Similarly, we can deduce that

$$\begin{aligned}
II &\leq \int_0^1 |F_z(x, u_0(x) + \hat{y}, \nabla u_0(x) + z + t(z - \hat{z})) \cdot (z - \hat{z})| dt + |F_z(x) \cdot (z - \hat{z})| \\
&\leq C(\hat{y})(1 + |z|^{p-1} + |z - \hat{z}|^{p-1})|z - \hat{z}| + c|z - \hat{z}| \\
&\leq C(y, \hat{y})(|y| + |\hat{y}| + |z| + |\hat{z}| + |z|^{p-1} + |\hat{z}|^{p-1})|z - \hat{z}|.
\end{aligned}$$

This concludes the proof of the Lemma, after using the definition of A_p . \square

We next proceed to prove Theorem 4.4 for the case in which Ω is itself a polytope. With the purpose of presenting more clearly the ideas behind this result, we refer the reader to Section 5 for the more technical part involving domains locally diffeomorphic to a polytope:

Proof of Theorem 4.4 (when Ω is a polytope). Let ω and C be functions as in (a)-(b) from Lemma 4.5. Take $\varepsilon < \frac{c_0}{8}$, with $c_0 > 0$ as in assumptions (III) and (IV)_k. For such ε , we take $R = R_\varepsilon$, $c_\varepsilon > 0$ given by Lemma 4.5 and we consider, for an arbitrary $x_0 \in \overline{\Omega}$, $\varphi \in \overline{\text{Var}(\Omega(x_0, R), \mathbb{R}^N)^{W^{1,p}}}$ satisfying $\|\varphi\|_{L^\infty} < \delta$, where $\delta > 0$ is such that $c_\varepsilon \delta C(\delta) < \frac{c_0}{4}$ and $c\omega(\delta) < \frac{c_0}{8}$. Then, Lemma 4.5 implies that

$$(4.3) \quad G(x_0, 0, \nabla \varphi(x)) - G(x, \varphi(x), \nabla \varphi(x)) \leq C|\varphi(x)|^2 + \frac{c_0}{2}|S(\nabla \varphi(x))|^2$$

for every $x \in \Omega(x_0, R)$, where $C > 0$ is a constant satisfying $c_\varepsilon C(\delta) \leq C$ for the locally bounded function $C(y)$.

It is now straightforward to conclude the proof. Note that the quasiconvexity condition (III) remains valid if B is replaced by $\Omega(x_0, R)$ and, by Lemma 2.12, the quasiconvexity condition (IV)_k also remains valid for any standard $(d - k)$ -boundary region. Of course, this is also true for the strong quasiconvexity conditions at hand. In particular, if the set $\Omega(x_0, R)$ is appropriate for one of

the quasiconvexity conditions (III) or (IV)_k then also $\varphi \in \overline{\text{Var}(\Omega(x_0, R), \mathbb{R}^N)^{W^{1,p}}}$ is an appropriate test function by Remark 2.10 and we obtain that

$$\begin{aligned} c_0 \int_{\Omega(x_0, R)} |S(\nabla \varphi(x))|^2 dx &\leq \int_{\Omega(x_0, R)} G(x_0, 0, \nabla \varphi(x)) dx \\ &\leq \int_{\Omega(x_0, R)} \left(G(x, \varphi(x), \nabla \varphi(x)) + \frac{c_0}{2} |S(\nabla \varphi(x))|^2 \right) dx \\ &\quad + C \int_{\Omega(x_0, R)} |\varphi(x)|^2 dx. \end{aligned}$$

Since u_0 satisfies the Euler-Lagrange equations, we deduce that

$$\begin{aligned} &\frac{c_0}{2} \int_{\Omega(x_0, R)} (|S(\nabla \varphi(x))|^2 - C|\varphi(x)|^2) dx \\ &\leq \int_{\Omega(x_0, R)} (F(x, u_0(x) + \varphi(x), \nabla u_0(x) + \nabla \varphi(x)) - F(x)) dx, \end{aligned}$$

which is the required inequality. Hence, we only need to show that for any $x_0 \in \overline{\Omega}$, the set $\Omega(x_0, R)$ is appropriate for one of the quasiconvexity conditions. However, if $x_0 \in \Omega$ or $x_0 \in \Gamma_D$ and $\Omega(x_0, R)$ does not intersect Γ_N it suffices to consider condition (III) (quasiconvexity in the interior). If $\Omega(x_0, R)$ intersects Γ_N (so in particular if $x_0 \in \Gamma_N$) then we pick the (by assumption) unique lowest dimensional boundary in the set $\Omega(x_0, R)$ and apply the relevant quasiconvexity condition. This is because, if the lowest dimensional boundary intersecting the set $\Omega(x_0, R)$ has dimension $d - k$ and is a portion of some $(d - k)$ -facelike boundary of Ω , then $\Omega(x_0, R)$ is a standard $(d - k)$ -boundary region with the corresponding normals and, by Remark 2.10, any $\varphi \in \overline{\text{Var}(\Omega(x_0, R), \mathbb{R}^N)^{W^{1,p}}}$ is an appropriate test function for quasiconvexity at that $(d - k)$ -facelike boundary. \square

Remark 4.6. Let $\Omega_Q(x_0, R) := \Omega \cap Q(x_0, R)$, where $Q(x_0, R)$ is a cube centred at x_0 , with sides parallel to the coordinate axes and side length $2R$. Then, it is easy to see that, for a function $\varphi \in \overline{\text{Var}(\Omega_Q(x_0, R), \mathbb{R}^N)^{W^{1,p}}}$, we can assign the value of 0 in $\Omega(x_0, 2R) \setminus \Omega_Q(x_0, R)$ and hence assume that $\varphi \in \overline{\text{Var}(\Omega(x_0, 2R), \mathbb{R}^N)^{W^{1,p}}}$. Therefore, Theorem 4.4 still remains valid if we exchange $\Omega(x_0, R)$ by $\Omega_Q(x_0, R)$ in the statement.

A fundamental tool in the proof of the Sufficiency Theorem (Theorem 4.2) is a decomposition result that finds its origins in the Decomposition Lemma established by Kristensen [Kri94, Kri99] and, by other means, by Fonseca-Müller-Pedregal in [FMP98]. This result allows, up to subsequences, the splitting of a weakly converging sequence into an oscillating and a concentrating part. We enunciate here Theorem 4.7 concerning a variant of the aforementioned Decomposition Lemma that enables us to split simultaneously the normalisations in $W^{1,2}$ and $W^{1,p}$, respectively, of a given sequence. This version of the Decomposition Lemma was established in [GM09], where its proof can be found. We remark that this is based in the Lipschitz truncation strategy followed in [FMP98], while Kristensen's proof uses the Helmholtz Decomposition Theorem.

Theorem 4.7. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain and $p \in [2, \infty)$. Let $(\psi_k) \subseteq \text{Var}(\mathcal{A})$ such that $\psi_k \rightharpoonup \psi$ in $W^{1,2}(\Omega, \mathbb{R}^N)$ and assume that (η_k) is a sequence in $(0, 1]$ such that $\eta_k \psi_k$ is bounded in $W^{1,p}(\Omega, \mathbb{R}^N)$. If $p = 2$, assume that $\eta_k = 1$. Further, suppose that $\alpha_k > 0$, $\alpha_k \rightarrow 0$ and that $\alpha_k \psi_k \rightarrow 0$ uniformly in Ω . Then, there exist a subsequence of (ψ_k) (not relabelled) and

sequences $(g_k) \subseteq -\psi + W^{1,\infty}(\Omega, \mathbb{R}^N)$, $(b_k) \subseteq W^{1,\infty}(\Omega, \mathbb{R}^N)$ with $\psi + g_k(x) = b_k(x) = 0$ for $x \in \Gamma_D$ such that

- (a) $g_k \rightharpoonup 0$ and $b_k \rightharpoonup 0$ in $W^{1,2}(\Omega, \mathbb{R}^N)$;
- (b) $(|\nabla g_k|^2)$ is equiintegrable;
- (c) $\nabla b_k \rightarrow 0$ in measure;
- (d) $\alpha_k(\psi + g_k) \rightarrow 0$ and $\alpha_k b_k \rightarrow 0$ uniformly in Ω and
- (e) $\psi_k = \psi + g_k + b_k$.

In addition, (g_k) and (b_k) can be chosen such that, for a subsequence of (η_k) ,

- (a') $\eta_k g_k \rightharpoonup 0$ and $\eta_k b_k \rightharpoonup 0$ in $W^{1,p}(\Omega, \mathbb{R}^N)$;
- (b') $(|\eta_k \nabla g_k|^p)$ is equiintegrable and
- (c') $\eta_k \nabla b_k \rightarrow 0$ in measure.

We note that since $b_k \in W^{1,\infty}(\Omega, \mathbb{R}^N)$ and $b_k = 0$ on Γ_D , then also $b_k \in \overline{\text{Var}(\mathcal{A})}^{W^{1,p}}$. The following lemma will also play a crucial role in the proof of Theorem 4.2 and, though simple, we present its proof here.

Lemma 4.8. *Let $F: \mathbb{R}^{N \times d} \rightarrow \mathbb{R}$ and $z_0 \in \mathbb{R}^{N \times d}$. Suppose that F is of class C^2 in a neighbourhood of z_0 and that it is strongly quasiconvex (in the interior) at z_0 , i.e. for all $\varphi \in C_c^1(B, \mathbb{R}^N)$,*

$$\int_B [F(z_0 + \nabla \varphi) - F(z_0)] \, dx \geq c_0 \int_B |S(\nabla \varphi)|^2 \, dx.$$

Then, for every $z_0 \in \mathbb{R}^{N \times d}$ and every $\varphi \in C_c^1(B, \mathbb{R}^N)$, it holds that

$$2c_0 \int_{\Omega} |\nabla \varphi|^2 \, dx \leq \int_{\Omega} F_{zz}(z_0) \nabla \varphi \cdot \nabla \varphi \, dx,$$

i.e. $F_{zz}(z_0)[\cdot, \cdot] - 2c_0|\cdot|^2$ is quasiconvex.

Proof. By the quasiconvexity at z_0 it follows that, for every $\varphi \in C_c^1(B, \mathbb{R}^N)$, $t = 0$ minimises the real valued function

$$(4.4) \quad J(t) := \int_B [F(z_0 + t \nabla \varphi) - F(z_0) - c_0 |S(t \nabla \varphi)|^2] \, dx.$$

Hence,

$$(4.5) \quad 0 \leq J''(0) = \int_B [F_{zz}(z_0) \nabla \varphi \cdot \nabla \varphi - 2c_0 |\nabla \varphi|^2] \, dx.$$

□

Remark 4.9. For F and u_0 as in Theorem 4.2, Lemma 4.8 says that condition (III) on the strong quasiconvexity in the interior implies that, for each $x \in \overline{\Omega}$ and any $a(x) \in \mathbb{R}^{N \times d}$, the function $H(x, z) = F_{zz}(x)z \cdot z - 2c_0|z - a(x)|^2$ is quasiconvex at every $z \in \mathbb{R}^{N \times d}$.

The remainder of this Section is devoted to the proof of the Sufficiency Theorem (Theorem 4.2).

Proof of Theorem 4.2. We prove the result arguing by contradiction. Suppose that the theorem does not hold. Then, we can find a sequence $(\varphi_k) \subseteq \text{Var}(\Omega, \mathbb{R}^N)$ such that $\|\varphi_k\|_{L^\infty(\Omega, \mathbb{R}^N)} \rightarrow 0$ and

$$(4.6) \quad \int_{\Omega} F(x, u_0(x) + \varphi_k(x), \nabla u_0(x) + \nabla \varphi_k(x)) \, dx < \int_{\Omega} F(x) \, dx$$

for all $k \in \mathbb{N}$. As in Lemma 4.5, we use Taylor's Theorem and define

$$\begin{aligned} G(x, y, z) &:= F(x, u_0(x) + y, \nabla u_0(x) + z) - F(x) - F_y(x) \cdot y - F_z(x) \cdot z \\ &= \int_0^1 (1-t) L(x, u_0(x) + ty, \nabla u_0(x) + tz) [(y, z), (y, z)] dt, \end{aligned}$$

where

$$L(x, v, w) [(y, z), (y, z)] = F_{yy}(x, v, w) y \cdot y + 2F_{yz}(x, v, w) y \cdot z + F_{zz}(x, v, w) z \cdot z.$$

Note that, since u_0 is an F -extremal, for every $k \in \mathbb{N}$ it holds that

$$\begin{aligned} & \int_{\Omega} G(x, \varphi_k, \nabla \varphi_k) dx \\ &= \int_{\Omega} \int_0^1 (1-t) L(x, u_0(x) + t\varphi_k(x), \nabla u_0(x) + t\nabla \varphi_k(x)) [(\varphi_k, \nabla \varphi_k), (\varphi_k, \nabla \varphi_k)] dt dx \\ (4.7) \quad &= \int_{\Omega} (F(x, u_0(x) + \varphi_k(x), \nabla u_0(x) + \nabla \varphi_k(x)) - F(x)) dx < 0. \end{aligned}$$

This inequality suggests the underlying idea of the proof which is to exploit the strong positivity of the second variation to obtain a contradiction. Indeed, by a normalisation argument we show that one can construct a sequence of variations (ψ_k) suitable for this purpose.

We remark that, for simplicity, we will use the notation $\Omega(x, r) := \Omega \cap Q(x, r)$, where $Q(x, r)$ is the cube with side length $2r$ rather than the ball $B(x, r)$. Note that, due to Remark 4.6, Theorem 4.4 remains valid. We divide the proof into several steps.

Step 0. In this preliminary step we establish that we can cover Ω by a finite sequence of cubes on each of which we can apply one of the interior or boundary quasiconvexity conditions. Furthermore, we construct this cover so that the radius of each of its elements lies within a given subset \mathcal{R} of $(0, \infty)$ with the property that $(0, \infty) - \mathcal{R}$ is countable.

Let (x_j) be an enumeration of \mathbb{Q}^d . The aim is to show that, if $R > 0$ is given by Theorem 4.4, we can find a sequence of radii $(r_j) \subseteq (0, \frac{R}{2})$, such that, for any $s_j \in [r_j, 2r_j]$, if a cube $\overline{Q(x_j, s_j)}$ intersects Γ_N , and if $d - k$ is the lowest dimension of the faces intersected by $\overline{Q(x_j, s_j)}$, then it intersects exactly one such $(d - k)$ -face. Furthermore, we wish to choose a finite set of indices J and (r_j) as above so that

$$\Omega \subseteq \bigcup_{j \in J} \overline{Q(x_j, r_j)}$$

and the open cubes $(Q(x_j, r_j))_{j \in J}$ are pairwise disjoint.

Proof of Step 0. We construct the cover as follows. Let $\mathcal{R} \subseteq (0, \infty)$ be fixed with $(0, \infty) - \mathcal{R}$ countable.¹ Observe that the set

$$\left\{ r \in \mathcal{R} : \text{exists } k \in \mathbb{N} \text{ with } \frac{r}{k} \notin \mathcal{R} \right\}$$

is countable. Therefore, without loss of generality we can assume that, for all $r \in \mathcal{R}$ and for all $k \in \mathbb{N}$, $\frac{r}{k} \in \mathcal{R}$.

Next, consider $R > 0$ provided by Theorem 4.4 and note that, by Definition 2.6, Ω can only have a finite number of 0-facelike points. Therefore, we may choose $\tilde{r} \in \mathcal{R} \cap (0, R)$ with the property that the cubes of the form $\overline{Q(x_j, \tilde{r})}$ and $\overline{Q(x_j, \frac{\tilde{r}}{2})}$ contain at most one 0-facelike point.

¹The need to consider radii only in such a set will become evident in the following steps of the proof.

We proceed to consider a grid of cubes with centre in \mathbb{Q}^d and radius $\frac{\tilde{r}}{2}$ such that

$$\Omega \subseteq \bigcup_{j \in \tilde{J}} \overline{Q\left(x_j, \frac{\tilde{r}}{2}\right)}.$$

Denoting by $Q(x_{j_1}, \tilde{r}/2), \dots, Q(x_{j_m}, \tilde{r}/2)$ the cubes that contain precisely one vertex of Ω , we observe that the connected components of $\partial\Omega - \bigcup_{i=1}^m Q(x_{j_i}, \tilde{r}/2)$ are pairwise disjoint compact sets. Hence, for $j \in J - \{j_1, \dots, j_m\}$ we consider the cubes $\overline{Q\left(x_j, \frac{\tilde{r}}{2}\right)}$ that intersect more than one 1-face and split them into smaller cubes of some radius $\frac{\tilde{r}}{2k}$, with $k \in \mathbb{N}$, in such a way that both $\overline{Q\left(x_j, \frac{\tilde{r}}{2k}\right)}$ and $\overline{Q\left(x_j, \frac{\tilde{r}}{k}\right)}$ intersect at most one 1-face.

Having covered the 0- and the 1-faces by closed cubes that intersect at most one face of minimum dimension, we consider the complement in $\partial\Omega$ of the union of such cubes and we repeat the process for the 2-faces. We proceed inductively until each remaining cube of the form $\overline{Q\left(x_j, \frac{\tilde{r}}{2k_j}\right)}$ intersects at most one $(d-k)$ -face, if $d-k$ is the minimum dimension of the faces that the given cube intersects.

We call r_j the radius of each cube $\overline{Q\left(x_j, \frac{\tilde{r}}{2k_j}\right)}$ that remained in our cover and we denote by J the finite set of indices with the property that $j \in J$ if, and only if, $\overline{Q\left(x_j, r_j\right)}$ belongs to the cover. This concludes the proof of Step 0.

Step 1. We now prove an inequality which is used repeatedly in the proof and is a direct consequence of Theorem 4.4.

We claim that there exists $\delta > 0$ such that, if $(Q(x_j, r_j))_{j \in J}$ is as in Step 0, then

$$\begin{aligned} & \frac{c_0}{2} \int_{\Omega} |S(\nabla \varphi(x))|^2 dx - c \sum_{j \in J} \int_{\Omega(x_j, s_j) - \Omega(x_j, r_j)} \left(|S(\nabla \varphi(x))|^2 + \left| S\left(\frac{\varphi(x)}{s_j - r_j}\right) \right|^2 \right) dx \\ & \leq \int_{\Omega} (F(x, u_0(x) + \varphi(x), \nabla u_0(x) + \nabla \varphi(x)) - F(x) + |\varphi(x)|^2) dx \\ (4.8) \quad & = \int_{\Omega} (G(x, \varphi(x), \nabla \varphi(x)) + |\varphi(x)|^2) dx \end{aligned}$$

for all $s_j \in (r_j, 2r_j)$ and all $\varphi \in \overline{\text{Var}(\Omega, \mathbb{R}^N)^{W^{1,p}}}$ satisfying $\|\varphi\|_{L^\infty(\Omega, \mathbb{R}^N)} < \delta$.

Proof of Step 1. By Theorem 4.4, there exist $R > 0$ and $\delta > 0$ such that, for all $x_0 \in \overline{\Omega}$,

$$\begin{aligned} & \frac{c_0}{2} \int_{\Omega(x_0, R)} (|S(\nabla \varphi(x))|^2 - C|\varphi(x)|^2) dx \\ (4.9) \quad & \leq \int_{\Omega(x_0, R)} (F(x, u_0(x) + \varphi(x), \nabla u_0(x) + \nabla \varphi(x)) - F(x)) dx \end{aligned}$$

for all $\varphi \in \overline{\text{Var}(\Omega(x_0, R), \mathbb{R}^N)^{W^{1,p}}}$ with $\|\varphi\|_\infty < \delta$ whenever $\Omega(x_0, R) \cap \Gamma_N \neq \emptyset$, provided $\Omega(x_0, R)$ intersects precisely one face of lowest dimension. Next, consider the cover for Ω built in Step 0 and, for each $j \in J$ and each $s_j \in (r_j, 2r_j)$, take cut-off functions $\rho_j \in C_c^1(Q(x_j, s_j))$ such that $\mathbb{1}_{Q(x_j, r_j)} \leq \rho_j \leq \mathbb{1}_{Q(x_j, s_j)}$ and $|\nabla \rho_j| \leq \frac{2}{s_j - r_j}$. Note that the cubes $Q(x_j, s_j)$ have bounded overlap since, when $s_j < 2r_j$, $Q(x_j, s_j)$ will intersect at most $3^d - 1$ other such cubes.

Additionally, if $\varphi \in \overline{\text{Var}(\Omega, \mathbb{R}^N)^{W^{1,p}}}$, then $\rho_j \varphi \in \text{Var}(\Omega(x_j, s_j), \mathbb{R}^N)^{W^{1,p}}$ and, by (4.9),

$$\begin{aligned} \int_{\Omega(x_j, s_j)} \frac{c_0}{2} |S(\nabla(\rho_j \varphi))|^2 dx &\leq \int_{\Omega(x_j, s_j)} [F(x, u_0 + \rho_j \varphi, \nabla u_0 + \nabla(\rho_j \varphi)) - F(x)] dx \\ &\quad + \int_{\Omega(x_j, s_j)} C |\rho_j \varphi|^2 dx. \end{aligned}$$

Since u_0 is an F -extremal, this also implies that

$$\frac{c_0}{2} \int_{\Omega(x_j, s_j)} |S(\nabla(\rho_j \varphi))|^2 dx \leq \int_{\Omega(x_j, s_j)} [G(x, \rho_j \varphi, \nabla(\rho_j \varphi)) + C |\rho_j \varphi|^2] dx.$$

But $\rho_j = 1$ on $Q(x_j, r_j)$ and, hence,

$$\begin{aligned} &\frac{c_0}{2} \int_{\Omega(x_j, r_j)} |S(\nabla \varphi)|^2 dx + \frac{c_0}{2} \int_{\Omega(x_j, s_j) - \Omega(x_j, r_j)} |S(\nabla(\rho_j \varphi))|^2 dx \\ &\leq \int_{\Omega(x_j, r_j)} [G(x, \varphi, \nabla \varphi) + C |\varphi|^2] dx + \int_{\Omega(x_j, s_j) - \Omega(x_j, r_j)} [G(x, \rho_j \varphi, \nabla(\rho_j \varphi)) + C |\rho_j \varphi|^2] dx. \end{aligned}$$

By Lemma 4.5 (a), we find that for each $x \in \overline{\Omega}$, $y \in \mathbb{R}^N$ and $z \in \mathbb{R}^{N \times d}$, the function G satisfies

$$(4.10) \quad |G(x, y, z)| \leq C(y) (|y|^2 + |y|^p + |z|^2 + |z|^p) = C(y) (|S(y)|^2 + |S(z)|^2)$$

for some locally bounded $C(y)$. Noting that φ is bounded, using (4.10) and after adding up the previous inequalities over j , we obtain

$$\begin{aligned} &\frac{c_0}{2} \int_{\Omega} |S(\nabla \varphi)|^2 dx + \frac{c_0}{2} \sum_{j \in J} \int_{\Omega(x_j, s_j) - \Omega(x_j, r_j)} |S(\nabla(\rho_j \varphi))|^2 dx \\ &\leq \int_{\Omega} (F(x, u_0 + \varphi, \nabla u_0 + \nabla \varphi) - F(x) + C |\varphi|^2) dx \\ &\quad + c \sum_{j \in J} \int_{\Omega(x_j, s_j) - \Omega(x_j, r_j)} [|S(\rho_j \varphi)|^2 + |S(\nabla(\rho_j \varphi))|^2 + C |\rho_j \varphi|^2] dx. \\ &\leq \int_{\Omega} (F(x, u_0 + \varphi, \nabla u_0 + \nabla \varphi) - F(x) + C |\varphi|^2) dx \\ &\quad + c \sum_{j \in J} \int_{\Omega(x_j, s_j) - \Omega(x_j, r_j)} \left[|S(\varphi)|^2 + |S(\nabla \varphi)|^2 + \left| S \left(\frac{\varphi}{s_j - r_j} \right) \right|^2 + C |\varphi|^2 \right] dx, \end{aligned}$$

since $|\rho_j| \leq 1$. Hence, (4.8) follows because $0 \leq s_j - r_j \leq 1$ and $|\xi|^2 \leq |S(\xi)|^2$.

Step 2. We next use the coercivity assumption [H2] to reduce the problem to the case of $W^{1,p}$ -local minimisers.

Let $\gamma_k := \|S(\nabla \varphi_k)\|_{L^2}$, $\alpha_k := \|\nabla \varphi_k\|_{L^2}$ and $\beta_k := (2|\Omega|)^{\frac{1}{2} - \frac{1}{p}} \|\nabla \varphi_k\|_{L^p}$. We claim that $\gamma_k \rightarrow 0$ and consequently, $\alpha_k \rightarrow 0$ and $\beta_k \rightarrow 0$. Moreover, the sequence of variations (φ_k) is such that

$$(4.11) \quad 0 \leq \sup_{k \in \mathbb{N}} \frac{\beta_k^p}{\alpha_k^2} = \Lambda < \infty$$

for some real number $\Lambda > 0$.

Proof of Step 2. By virtue of assumption [H2] and (4.6), (φ_k) is uniformly bounded in $W^{1,p}(\Omega, \mathbb{R}^N)$ and must therefore converge weakly to 0 (since it converges strongly to 0 in $L^\infty(\Omega, \mathbb{R}^N)$).

Note that $\gamma_k > 0$ for all $k \in \mathbb{N}$ and (γ_k) is a bounded sequence because $p \geq 2$. Therefore, up to a subsequence that we do not relabel, $\gamma_k \rightarrow \gamma \geq 0$. To reach a contradiction, suppose that $\gamma > 0$. Considering a further subsequence, we may also assume that $|S(\nabla \varphi_k)|^2 \mathcal{L}^d \xrightarrow{*} \mu$ in $C^0(\overline{\Omega})^* \cong \mathcal{M}(\overline{\Omega})$, where \mathcal{L}^d denotes the d -dimensional Lebesgue measure.

We take $r_j \in (0, R)$ and the grid from Step 0 so that $\mu(\bigcup_{j \in J} (\partial(Q(x_j, r_j)) \cap \overline{\Omega})) = 0$. This is possible because, for a given x_0 , only a countable amount of cubes can satisfy $\mu(\partial Q(x_0, r)) > 0$. To prove this, observe that for any $k \in \mathbb{N}$,

$$\mathcal{A}_k := \left\{ r \in (0, R) : \mu(\partial Q(x_0, r)) > \frac{1}{k} \right\}$$

is a pairwise disjoint collection of subsets of $Q(x_0, R)$. Since μ is σ -additive and $\mu(Q(x_0, R))$ is a positive real number, this implies that \mathcal{A}_k is finite for every $k \in \mathbb{N}$. Hence, the set of real numbers

$$(4.12) \quad \{r \in (0, \infty) : \mu(\partial Q(x_0, r)) > 0\} = \bigcup_{k \in \mathbb{N}} \mathcal{A}_k$$

is at most countable. In the notation of Step 0, we remark that \mathcal{R} is a subset of $(0, R) - \bigcup_{k \in \mathbb{N}} \mathcal{A}_k$. Now observe that, for $r_j < s_j < 2r_j$, we obtain from inequality (4.8) in Step 1 applied to $\varphi = \varphi_k$, that

$$\begin{aligned} & \frac{c_0}{2} \int_{\Omega} |S(\nabla \varphi_k)|^2 dx - c \sum_{j \in J} \int_{\Omega(x_j, s_j) - \Omega(x_j, r_j)} \left(|S(\nabla \varphi_k)|^2 + \left| S\left(\frac{\varphi_k}{s_j - r_j}\right) \right|^2 \right) dx \\ & \leq \int_{\Omega} [F(x, u_0 + \varphi_k, \nabla u_0 + \nabla \varphi_k) - F(x) + C|\varphi_k|^2] dx. \end{aligned}$$

Recall that $\varphi_k \rightarrow 0$ in $L^\infty(\Omega, \mathbb{R}^N)$, implying that $S(\varphi_k) \rightarrow 0$ in $L^2(\Omega, \mathbb{R}^N)$. Hence,

$$\frac{c_0}{2} \gamma^2 - c \mu \left(\overline{\Omega} \cap \bigcup_{j \in J} (\overline{Q(x_j, s_j)} - Q(x_j, r_j)) \right) \leq 0$$

and, letting $s_j \searrow r_j$ in the above expression, we infer that

$$0 < \frac{c_0}{2} \gamma^2 = \frac{c_2}{2} \gamma^2 - c \mu \left(\overline{\Omega} \cap \bigcup_{j \in J} \partial(Q(x_j, r_j)) \right) \leq 0.$$

This is a contradiction establishing that $\gamma_k \rightarrow 0$. Regarding the boundedness of β_k^p / α_k^2 , this is trivial if $p = 2$. If $p > 2$, from the coercivity assumption (H2) applied to φ_k it follows, after dividing by α_k^2 , that for every $k \in \mathbb{N}$,

$$c_1 \frac{\beta_k^p}{\alpha_k^2} - c_2 \leq \alpha_k^{-2} \int_{\Omega} [F(x, u_0(x) + \varphi_k(x), \nabla u_0(x) + \nabla \varphi_k(x)) - F(x)] dx < 0.$$

This proves that the sequence $\left(\frac{\beta_k^p}{\alpha_k^2} \right)$ is bounded and the claim follows. We remark that assumption [H2] was required precisely to reduce the problem to that of $W^{1,p}$ -local minimisers.

Step 3. In this step we define our normalised sequence of variations and prove the following main inequality which will enable us to reach a contradiction:

$$(4.13) \quad \frac{1}{2} \int_{\Omega} \left[F_{yy}(x) \psi(x) \cdot \psi(x) + 2F_{yz}(x) \psi(x) \cdot \nabla \psi(x) + \int F_{zz}(x) z \cdot z dv_x(z) \right] dx \leq 0.$$

Here, ψ is the weak limit of (a subsequence) of $\psi_k := \alpha_k^{-1} \varphi_k \in \text{Var}(\Omega, \mathbb{R}^N)$ which is certainly bounded in $W^{1,2}$ and $\nu = (\nu_x)_{x \in \Omega}$ is the gradient p -Young measure such that (also up to a subsequence) $\nabla \psi_k \xrightarrow{Y} \nu$.

Proof of Step 3. We note that the left hand side of (4.13) resembles the second variation at u_0 evaluated at $\psi \in \overline{\text{Var}(\mathcal{A})}^{W^{1,2}}$. Furthermore, we observe that considering the normalisation ψ_k , instead of merely the sequence φ_k , is necessary to contradict the positivity of the second variation precisely because φ_k converges strongly to 0 in $W^{1,p}(\Omega, \mathbb{R}^N)$ and, hence, the analogous left hand side of (4.13) would also be 0.

On the other hand, the sequence $(\nabla \psi_k)$ may not be 2-equintegrable and, therefore, passing to the Young measure limit on the right hand side of (4.8) is not possible. We therefore decompose the sequence into an oscillating and a concentrating part using Lemma 4.7.

To this end, let $\eta_k := \frac{\alpha_k}{\beta_k}$ and note that, by Hölder's inequality,

$$\alpha_k = \|\nabla \varphi_k\|_{L^2} \leq (2|\Omega|)^{\frac{1}{2} - \frac{1}{p}} \|\nabla \varphi_k\|_{L^p} = \beta_k.$$

Therefore, $\eta_k = \frac{\alpha_k}{\beta_k} \in (0, 1]$ for every $k \in \mathbb{N}$. Also, it is clear that $\alpha_k \psi_k = \varphi_k \rightarrow 0$ uniformly and, to apply the Decomposition Lemma, we need only show that $(\eta_k \psi_k)$ is bounded in $W^{1,p}(\Omega, \mathbb{R}^N)$. But $\int_{\Omega} |\eta_k \nabla \psi_k|^p = \beta_k^{-p} \int_{\Omega} |\nabla \varphi_k|^p = 1$. Then, up to a subsequence which we do not relabel, there exist sequences $(g_k) \subseteq -\psi + W^{1,\infty}(\Omega, \mathbb{R}^N)$ and $(b_k) \subseteq W^{1,\infty}(\Omega, \mathbb{R}^N)$ with $\psi + g_k = b_k = 0$ on Γ_D such that:

- $\psi_k = \psi + g_k + b_k$;
- $g_k \rightharpoonup 0$ and $b_k \rightharpoonup 0$ in $W^{1,2}(\Omega, \mathbb{R}^N)$;
- $\nabla b_k \rightarrow 0$ in measure
- $(|\nabla g_k|^2)$ and $(|\eta_k \nabla g_k|^p)$ are both equintegrable;
- $\alpha_k(\psi + g_k) \rightarrow 0$ and $\alpha_k b_k \rightarrow 0$ uniformly in Ω and
- $\eta_k(\psi + g_k) \rightharpoonup 0$ and $\eta_k b_k \rightharpoonup 0$ in $W^{1,p}(\Omega, \mathbb{R}^N)$.

Let us write

$$f_k(x) := \alpha_k^{-2} G(x, \alpha_k \psi_k, \alpha_k \nabla \psi_k) - \alpha_k^{-2} G(x, \alpha_k b_k, \alpha_k \nabla b_k),$$

so that, using $\alpha_k \psi_k = \varphi_k$ and (4.7), we obtain

$$\int_{\Omega} [f_k(x) + \alpha_k^{-2} G(x, \alpha_k b_k, \alpha_k \nabla b_k)] dx = \alpha_k^{-2} \int_{\Omega} G(x, \alpha_k \psi_k, \alpha_k \nabla \psi_k) dx < 0.$$

We aim to prove that

$$(4.14) \quad \begin{aligned} & \frac{1}{2} \int_{\Omega} \left[F_{yy}(x) \psi(x) \cdot \psi(x) + 2F_{yz}(x) \psi(x) \cdot \nabla \psi(x) + \int F_{zz}(x) z \cdot z d\nu_x(z) \right] dx \\ & \leq \liminf_k \int_{\Omega} f_k(x) dx \end{aligned}$$

and that

$$(4.15) \quad \liminf_k \alpha_k^{-2} \int_{\Omega} G(x, \alpha_k b_k, \alpha_k \nabla b_k) dx \geq 0,$$

which in turn establish (4.13).

We first show (4.15). Take $\varphi = \alpha_k b_k$ in inequality (4.8). Since u_0 is an F -extremal and $b_k \in \overline{\text{Var}(\Omega, \mathbb{R}^N)^{W^{1,p}}}$, after dividing by α_k^2 , we get that

$$\begin{aligned} & \frac{c_0}{2} \int_{\Omega} \left(|\nabla b_k|^2 + \alpha_k^{p-2} |\nabla b_k|^p \right) dx \\ & - c \sum_{j \in J} \int_{\Omega(x_j, s_j) - \Omega(x_j, r_j)} \left(|\nabla b_k|^2 + \alpha_k^{p-2} |\nabla b_k|^p + \frac{|b_k|^2}{(s_j - r_j)^2} + \alpha_k^{p-2} \frac{|b_k|^p}{(s_j - r_j)^p} \right) dx \\ & \leq \alpha_k^{-2} \int_{\Omega} (G(x, \alpha_k b_k, \alpha_k \nabla b_k) + |b_k|^2) dx \end{aligned}$$

for every r_j, s_j as in Step 0 such that $0 < r_j < s_j < 2r_j < 1 + r_j$. Notice that

$$\alpha_k^{p-2} (|b_k|^p + |\nabla b_k|^p) = \frac{\beta_k^p}{\alpha_k^2} \eta_k^p (|b_k|^p + |\nabla b_k|^p).$$

Since $\eta_k b_k \rightharpoonup 0$ in $W^{1,p}(\Omega, \mathbb{R}^N)$ and $\left(\frac{\beta_k^p}{\alpha_k^2}\right)$ is bounded according to (4.11), we deduce that, for a subsequence that we do not relabel, it also holds that

$$\alpha_k^{\frac{p-2}{p}} b_k = \beta_k \alpha_k^{-\frac{2}{p}} \eta_k b_k \rightharpoonup 0 \quad \text{in} \quad W^{1,p}(\Omega, \mathbb{R}^N).$$

We can now use this, and the fact that $b_k \rightharpoonup 0$ in $W^{1,2}(\Omega, \mathbb{R}^N)$, to proceed exactly as we did to prove that $\gamma_k \rightarrow 0$ and whereby conclude (4.15) since

$$\begin{aligned} & 0 \leq \frac{c_0}{2} \liminf_{k \rightarrow \infty} \int_{\Omega} |\nabla b_k|^2 + \alpha_k^{p-2} |\nabla b_k|^p dx \\ & = \liminf_{k \rightarrow \infty} \left[\frac{c_0}{2} \int_{\Omega} \left(|\nabla b_k|^2 + \alpha_k^{p-2} |\nabla b_k|^p \right) dx \right. \\ & \quad \left. - c \sum_{j \in J} \int_{\Omega(x_j, s_j) - \Omega(x_j, r_j)} \left(|\nabla b_k|^2 + \alpha_k^{p-2} |\nabla b_k|^p + \frac{|b_k|^2}{(s_j - r_j)^2} + \alpha_k^{p-2} \frac{|b_k|^p}{(s_j - r_j)^p} \right) dx \right] \\ & \leq \liminf_{k \rightarrow \infty} \int_{\Omega} (\alpha_k^{-2} G(x, \alpha_k b_k, \alpha_k \nabla b_k) + |b_k|^2) dx \\ & = \liminf_{k \rightarrow \infty} \int_{\Omega} \alpha_k^{-2} G(x, \alpha_k b_k, \alpha_k \nabla b_k) dx. \end{aligned}$$

Regarding (4.14), we first prove the equiintegrability of f_k . Note that by Lemma 4.5 (c), for every $y, \tilde{y} \in \mathbb{R}^N$, $z, \tilde{z} \in \mathbb{R}^{N \times d}$ and for every $x \in \overline{\Omega}$,

$$|G(x, y, z) - G(x, \tilde{y}, \tilde{z})| \leq C(y, \tilde{y}) [A_{p-1}(y, z, \tilde{y}, \tilde{z}) |z - \tilde{z}| + A_p(y, z, \tilde{y}, \tilde{z}) |y - \tilde{y}|].$$

Then, by Young's inequality, we obtain that for any $\varepsilon > 0$, there exists a constant C_ε such that

$$\begin{aligned}
 |f_k| &\leq C\alpha_k^{-1} (A_{p-1}(\alpha_k\psi_k, \alpha_k b_k, \alpha_k \nabla \psi_k, \alpha_k \nabla b_k) |\nabla \psi + \nabla g_k|) \\
 &\quad + C\alpha_k^{-1} (A_p(\alpha_k\psi_k, \alpha_k b_k, \alpha_k \nabla \psi_k, \alpha_k \nabla b_k)) |\psi + g_k| \\
 &\leq C \left(|\psi_k| + |b_k| + |\nabla \psi_k| + |\nabla b_k| + \alpha_k^{p-2} |\nabla \psi_k|^{p-1} + \alpha_k^{p-2} |\nabla b_k|^{p-1} \right) |\nabla \psi + \nabla g_k| \\
 &\quad + C \left(|\psi_k| + |b_k| + |\nabla \psi_k| + |\nabla b_k| + \alpha_k^{p-1} |\nabla \psi_k|^p + \alpha_k^{p-1} |\nabla b_k|^p \right) |\psi + g_k| \\
 &\leq \varepsilon C \left(|\psi_k|^2 + |b_k|^2 + |\nabla \psi_k|^2 + |\nabla b_k|^2 + \alpha_k^{p-2} |\nabla \psi_k|^p + \alpha_k^{p-2} |\nabla b_k|^p \right) \\
 &\quad + C_\varepsilon \left(|\nabla \psi + \nabla g_k|^2 + \alpha_k^{p-2} |\nabla \psi + \nabla g_k|^p \right) \\
 &\quad + \varepsilon C \left(|\psi_k|^2 + |b_k|^2 + |\nabla \psi_k|^2 + |\nabla b_k|^2 \right) \\
 &\quad + C_\varepsilon |\psi + g_k|^2 + \left(\alpha_k^{p-2} |\nabla \psi_k|^p + \alpha_k^{p-2} |\nabla b_k|^p \right) |\alpha_k(\psi + g_k)| =: \sum_{i=1}^5 e_i(x).
 \end{aligned}$$

Above, C is a constant (varying from line to line) since $\alpha_k\psi_k$ and $\alpha_k b_k$ are uniformly bounded. Observe that ψ_k, b_k are both bounded in $W^{1,2}(\Omega, \mathbb{R}^N)$ and, additionally,

$$\alpha_k^{p-2} |\nabla \psi_k|^p = \frac{\beta_k^p}{\alpha_k^2} |\eta_k \nabla \psi_k|^p$$

which is bounded in $L^1(\Omega, \mathbb{R}^N)$ since $\left(\frac{\beta_k^p}{\alpha_k^2}\right)$ is a bounded sequence by (4.11) and $(\eta_k \psi_k)$ is bounded in $W^{1,p}(\Omega, \mathbb{R}^N)$. Similarly, the same is true for $\alpha_k^{p-2} |\nabla b_k|^p$ by the boundedness of $(\eta_k b_k)$ in $W^{1,p}(\Omega, \mathbb{R}^N)$. Hence, for any set $A \subset \Omega$ we have

$$\int_A |f_k(x)| dx \leq \varepsilon C + \int_A e_2(x) dx + \int_A e_4(x) dx + \int_A e_5(x) dx.$$

Next note that by the Decomposition Lemma $\alpha_k(\psi + g_k) \rightarrow 0$ uniformly and hence e_5 is equiintegrable. Also, by the boundedness of $\left(\frac{\beta_k^p}{\alpha_k^2}\right)$ and the fact that

$$\alpha_k^{p-2} |\nabla g_k|^p = \frac{\beta_k^p}{\alpha_k^2} \eta_k^p |\nabla g_k|^p,$$

we deduce that $(\alpha_k^{p-2} |\nabla g_k|^p)$ is equiintegrable and, hence, so is e_2 . Finally, since $(|\psi + g_k|^2)$ is also equiintegrable, the same holds for (f_k) .

Now, let $\varepsilon > 0$. Since $(\nabla \psi_k)$ is measure-tight² and $\nabla b_k \rightarrow 0$ in measure, we can take $m_\varepsilon > 0$ large enough so that, for every $m \geq m_\varepsilon$,

$$\int_{\{|\nabla \psi_k| \geq m\} \cup \{|\nabla b_k| \geq m\}} |f_k(x)| dx < \varepsilon$$

for all $k \in \mathbb{N}$. Then, for all $m \geq m_\varepsilon$,

$$(4.16) \quad \int_{\{|\nabla \psi_k| < m\} \cap \{|\nabla b_k| < m\}} f_k(x) dx - \varepsilon < \int_{\Omega} f_k(x) dx.$$

Also note that ν_x , the Young measure generated by $(\nabla \psi_k)$, is a probability measure for a.e. $x \in \Omega$ and thus, by taking m_ε larger if necessary, we may assume that

$$\left| \int_{\Omega} \int_{\mathbb{R}^{N \times d}} F_{zz}(x) z \cdot z \mathbb{1}_{\mathbb{R}^{N \times d} \setminus \overline{B(0,m)}}(z) d\nu_x(z) dx \right| < \varepsilon \text{ for all } m \geq m_\varepsilon.$$

²A sequence (u_k) is measure-tight if $\lim_{t \rightarrow \infty} \sup_k \mathcal{L}^d(\{x \in \Omega : |u_k(x)| > t\}) = 0$.

Then, for $m \geq m_\varepsilon$,

$$\begin{aligned}
 & \int_{\Omega} \left[F_{yy}(x) \psi(x) \cdot \psi(x) + 2F_{yz}(x) \psi(x) \cdot \nabla \psi(x) + \int F_{zz}(x) z \cdot z \, d\nu_x(z) \right] dx \\
 & \leq \int_{\Omega} F_{yy}(x) \psi(x) \cdot \psi(x) + 2F_{yz}(x) \psi(x) \cdot \nabla \psi(x) dx \\
 (4.17) \quad & + \int_{\Omega} \int F_{zz}(x) z \cdot z \, \mathbb{1}_{B(0,m)}(z) \, d\nu_x(z) dx + \varepsilon.
 \end{aligned}$$

Next, consider the integrand $H: \overline{\Omega} \times \mathbb{R}^{N \times d} \rightarrow \mathbb{R}$ given by

$$H(x, z) := F_{zz}(x) z \cdot z \, \mathbb{1}_{B(0,m)}(z).$$

Note that $\mathbb{1}_{B(0,m)}(z)$ is lower semicontinuous because $B(0,m)$ is an open set. Hence, $H(x, \cdot)$ is lower semicontinuous for every $x \in \overline{\Omega}$ and, since $\nabla \psi_k$ generates the Young measure ν_x ,

$$\int_{\Omega} \int F_{zz}(x) z \cdot z \, \mathbb{1}_{B(0,m)}(z) \, d\nu_x(z) dx \leq \liminf_{k \rightarrow \infty} \int_{|\nabla \psi_k| < m} F_{zz}(x) \nabla \psi_k \cdot \nabla \psi_k dx.$$

Also, $\psi_k \rightarrow \psi$ strongly in $L^2(\Omega, \mathbb{R}^N)$ and $(\nabla \psi_k)$ is bounded in $L^2(\Omega, \mathbb{R}^{N \times d})$, so that the family $(F_{yy}(x) \psi_k \cdot \psi_k + 2F_{yz}(x) \psi_k \cdot \nabla \psi_k)_k$ is equiintegrable and, by Young measure representation,

$$\int_{\Omega} [F_{yy}(x) \psi \cdot \psi + 2F_{yz}(x) \psi \cdot \nabla \psi] dx = \lim_{k \rightarrow \infty} \int_{\Omega} [F_{yy}(x) \psi_k \cdot \psi_k + 2F_{yz}(x) \psi_k \cdot \nabla \psi_k] dx.$$

Combining the last two equations with (4.17), we obtain that for all $m \geq m_\varepsilon$,

$$\begin{aligned}
 & \int_{\Omega} \left[F_{yy}(x) \psi \cdot \psi + 2F_{yz}(x) \psi \cdot \nabla \psi + \int F_{zz}(x) z \cdot z \, d\nu_x(z) \right] dx \\
 & \leq \liminf_{k \rightarrow \infty} \int_{\{|\nabla \psi_k| < m\}} [F_{yy}(x) \psi_k \cdot \psi_k + 2F_{yz}(x) \psi_k \cdot \nabla \psi_k + F_{zz}(x) \nabla \psi_k \cdot \nabla \psi_k] dx \\
 & \quad + \lim_{k \rightarrow \infty} \int_{\{|\nabla \psi_k| \geq m\}} [F_{yy}(x) \psi_k \cdot \psi_k + 2F_{yz}(x) \psi_k \cdot \nabla \psi_k] dx + \varepsilon \\
 (4.18) \quad & \leq \liminf_{k \rightarrow \infty} \int_{\{|\nabla \psi_k| < m\}} [F_{yy}(x) \psi_k \cdot \psi_k + 2F_{yz}(x) \psi_k \cdot \nabla \psi_k + F_{zz}(x) \nabla \psi_k \cdot \nabla \psi_k] dx + 2\varepsilon.
 \end{aligned}$$

Note that the last inequality follows from the fact that $(\nabla \psi_k)$ is measure-tight, the equiintegrability of the sequence $(F_{yy}(x) \psi_k \cdot \psi_k + 2F_{yz}(x) \psi_k \cdot \nabla \psi_k)_k$, and by choosing m_ε larger if necessary.

We now claim that

$$\begin{aligned}
 & \frac{1}{2} \liminf_{k \rightarrow \infty} \int_{\{|\nabla \psi_k| < m\}} [F_{yy}(x) \psi_k \cdot \psi_k + 2F_{yz}(x) \psi_k \cdot \nabla \psi_k + F_{zz}(x) \nabla \psi_k \cdot \nabla \psi_k] dx \\
 (4.19) \quad & = \liminf_{k \rightarrow \infty} \int_{\{|\nabla \psi_k| < m\} \cap \{|\nabla b_k| < m\}} f_k(x) dx.
 \end{aligned}$$

Then, by (4.16), (4.18) and letting $\varepsilon \rightarrow 0$ (the dependence on m_ε will have been removed), we conclude (4.14), that is

$$\begin{aligned}
 & \frac{1}{2} \int_{\Omega} \left[F_{yy}(x) \psi(x) \cdot \psi(x) + 2F_{yz}(x) \psi(x) \cdot \nabla \psi(x) + \int F_{zz}(x) z \cdot z \, d\nu_x(z) \right] dx \\
 & \leq \liminf_{k \rightarrow \infty} \int_{\Omega} f_k(x) dx.
 \end{aligned}$$

To this end, let us use the notation

$$L(ty, tz)[y, z] = L(x, u_0(x) + ty, \nabla u_0(x) + tz)[(y, z), (y, z)]$$

and note that

$$\int_{\{|\nabla \psi_k| < m\} \cap \{|\nabla b_k| < m\}} f_k(x) dx = I_1^k + I_2^k + I_3^k + I_4^k,$$

where

$$\begin{aligned} I_1^k &= \int_{\Omega} \mathbb{1}_{\{|\nabla \psi_k| < m\} \cap \{|\nabla b_k| < m\}} \int_0^1 (1-t) [L(t\alpha_k \psi_k, t\alpha_k \nabla \psi_k) - L(0,0)] [\psi_k, \nabla \psi_k] dt dx; \\ I_2^k &= \frac{1}{2} \int_{\{|\nabla \psi_k| < m\}} L(0,0) [\psi_k, \nabla \psi_k] dx; \\ I_3^k &= -\frac{1}{2} \int_{\{|\nabla \psi_k| < m\}} L(0,0) [\psi_k, \nabla \psi_k] (1 - \mathbb{1}_{\{|\nabla b_k| < m\}}) dx; \\ I_4^k &= -\int_{\Omega} \mathbb{1}_{\{|\nabla \psi_k| < m\} \cap \{|\nabla b_k| < m\}} \int_0^1 (1-t) L(t\alpha_k b_k, t\alpha_k \nabla b_k) [b_k, \nabla b_k] dt dx. \end{aligned}$$

The term I_2^k is precisely the one appearing on the right-hand side of (4.19) and it thus suffices to prove that I_1^k , I_3^k and I_4^k all converge to 0 as $k \rightarrow \infty$. It is clear, by the Dominated Convergence Theorem, that since $\alpha_k \rightarrow 0$, $I_1^k \rightarrow 0$ as $k \rightarrow \infty$. Regarding the term I_4^k , note that the sequence of functions

$$L(t\alpha_k b_k, t\alpha_k \nabla b_k) [\nabla b_k, \nabla b_k] \mathbb{1}_{\{|\nabla b_k| < m\} \cap \{|\nabla \psi_k| < m\}}$$

is bounded in $L^\infty(\Omega)$ for all $t \in [0, 1]$ (recall $\alpha_k b_k \rightarrow 0$ uniformly) and, therefore, it is equiintegrable. In addition, this sequence converges to 0 in measure because $\nabla b_k \rightarrow 0$ in measure and, by [H0], all partial derivatives of second order of F are continuous. These two facts imply, by Vitali's Convergence Theorem, that $I_4^k \rightarrow 0$ as $k \rightarrow \infty$.

Furthermore, given that $\nabla b_k \rightarrow 0$ in measure and $(|\psi_k|)$ is equiintegrable, we also have that

$$|I_3^k| \leq cm \int_{\Omega} |\psi_k| (1 - \mathbb{1}_{\{|\nabla b_k| < m\}}) dx \rightarrow 0, \text{ as } k \rightarrow \infty.$$

Hence, (4.19) follows and, consequently, also (4.14). This concludes the proof of Step 3.

Step 4. We may now reach the desired contradiction and establish Theorem 4.2. Recall that (ψ_k) is a sequence of variations, uniformly bounded in $W^{1,2}(\Omega, \mathbb{R}^N)$, $\psi_k \rightharpoonup \psi$ in $W^{1,2}(\Omega, \mathbb{R}^N)$ and $\nabla \psi_k$ generates the Young measure $(\nu_x)_x$. By Lemma 4.8 (see also Remark 4.9), for any $x \in \Omega$, the function $H(x, z) = F_{zz}(x)z \cdot z - 2c_0|z - \bar{\nu}_x|^2$ is quasiconvex at every $z \in \mathbb{R}^{N \times d}$, where $\bar{\nu}_x = \nabla \psi(x)$ a.e. in Ω . In particular, since $(\nu_x)_x$ is a gradient Young measure and $H(x, \cdot)$ has quadratic growth, Jensen's inequality from the characterisation of gradient Young measures implies that for a.e. $x \in \Omega$,

$$(4.20) \quad \frac{1}{2} F_{zz}(x) \nabla \psi(x) \cdot \nabla \psi(x) + c_0 \int_{\mathbb{R}^{N \times d}} |z - \nabla \psi(x)|^2 d\nu_x(z) \leq \int_{\mathbb{R}^{N \times d}} F_{zz}(x) z \cdot z d\nu_x(z).$$

On the other hand, in Step 3 we showed that

$$\int_{\Omega} \left[F_{yy}(x) \psi \cdot \psi + 2F_{yz}(x) \psi \cdot \nabla \psi + \int F_{zz}(x) z \cdot z d\nu_x(z) \right] dx \leq 0$$

and, by (4.20), we may hence deduce that

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} [F_{yy}(x) \psi \cdot \psi + 2F_{yz}(x) \psi \cdot \nabla \psi + F_{zz}(x) \nabla \psi \cdot \nabla \psi] dx \\ & + c_0 \int_{\Omega} \int |z - \nabla \psi|^2 d\nu_x(z) dx \leq 0. \end{aligned}$$

But the second variation has been assumed strongly positive (see assumption (II) in Theorem 4.2), so that the above equation becomes

$$\frac{c_0}{2} \int_{\Omega} |\nabla \psi|^2 dx + c_0 \int_{\Omega} |z - \nabla \psi|^2 dv_x(z) dx \leq 0,$$

i.e. $\nabla \psi = 0$, $\psi = 0$ (by Poincaré's inequality) and $v_x = \delta_0$ a.e. in Ω . In particular, we infer that $\nabla \psi_k \rightarrow 0$ in $L^2(\Omega, \mathbb{R}^N)$ and that $\nabla \psi_k \rightarrow 0$ in measure (since the generated measure is an elementary Young measure).

To conclude the proof, we may now take $\varphi = \alpha_k \psi_k$ in inequality (4.8) and, after dividing by α_k^2 , we obtain that

$$\begin{aligned} & \frac{c_0}{2} \int_{\Omega} |\nabla \psi_k|^2 dx \\ & - c \sum_{j \in J} \int_{\Omega(x_j, s_j) - \Omega(x_j, r_j)} \left(|\nabla \psi_k|^2 + \alpha_k^{p-2} |\nabla \psi_k|^p + \frac{|\psi_k|^2}{(s_j - r_j)^2} + \alpha_k^{p-2} \frac{|\psi_k|^p}{(s_j - r_j)^p} \right) dx \\ & \leq \frac{c_0}{2} \int_{\Omega} \left(|\nabla \psi_k|^2 + \alpha_k^{p-2} |\nabla \psi_k|^p \right) dx \\ & - c \sum_{j \in J} \int_{\Omega(x_j, s_j) - \Omega(x_j, r_j)} \left(|\nabla \psi_k|^2 + \alpha_k^{p-2} |\nabla \psi_k|^p + \frac{|\psi_k|^2}{(s_j - r_j)^2} + \alpha_k^{p-2} \frac{|\psi_k|^p}{(s_j - r_j)^p} \right) dx \\ & < \int_{\Omega} |\psi_k|^2 dx \end{aligned}$$

for all $k \in \mathbb{N}$ and for all r_j, s_j as in Step 0 satisfying $0 < r_j < s_j < 2r_j < 1 + r_j$. Since

$$\alpha_k^{\frac{p-2}{p}} \psi_k = \beta_k \alpha_k^{-\frac{2}{p}} \eta_k \psi_k,$$

for a subsequence that we do not relabel, we may use (4.11) to further deduce that $\alpha_k^{\frac{p-2}{p}} \psi_k \rightarrow 0$ in $W^{1,p}(\Omega, \mathbb{R}^N)$. Arguing exactly as we did to prove that $\gamma_k \rightarrow 0$ as well as to prove inequality (4.15), we may now take the limit when $k \rightarrow \infty$ and use the property $\int_{\Omega} |\nabla \psi_k|^2 dx = 1$, to obtain that

$$0 < \frac{c_0}{2} \leq 0.$$

This contradiction concludes the proof of Theorem 4.2. \square

5. THE CASE OF DOMAINS LOCALLY DIFFEOMORPHIC TO A POLYTOPE

In this section we establish the proof of Theorem 4.4 for the general case in which Ω is locally diffeomorphic to a polytope. We show first the following technical result that complements the properties established in Lemma 4.5 regarding the linearisation of F .

Lemma 5.1. *Let $F: \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{N \times d} \rightarrow \mathbb{R}$ continuous and $u_0 \in C^1(\overline{\Omega}, \mathbb{R}^N)$ be such that [H0], [H1] and [UC] hold. Define $G: \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{N \times d} \rightarrow \mathbb{R}$ by*

$$\begin{aligned} G(x, y, z) &:= F(x, u_0(x) + y, \nabla u_0(x) + z) - F(x, u_0(x), \nabla u_0(x)) \\ &\quad - F_y(x, u_0(x), \nabla u_0(x)) \cdot y - F_z(x, u_0(x), \nabla u_0(x)) \cdot z \\ &= \int_0^1 (1-t) L(x, u_0(x) + ty, \nabla u_0(x) + tz) [(y, z), (y, z)], \end{aligned}$$

where the bilinear form $L(x, v, w)$ is given by

$$L(x, v, w)[(y, z), (\hat{y}, \hat{z})] := F_{yy}(x, v, w)y \cdot \hat{y} + F_{yz}(x, v, w)y \cdot \hat{z} + F_{yz}(x, v, w)\hat{y} \cdot \hat{z} + F_{zz}(x, v, w)\hat{z} \cdot \hat{z}.$$

For each $x \in \overline{\Omega}$, $y \in \mathbb{R}^N$ and $z \in \mathbb{R}^{N \times d}$, denoting by I_d the identity matrix in $\mathbb{R}^{d \times d}$, the function G satisfies the following:

- (a) For every $\varepsilon > 0$ there exists $\delta = \delta_\varepsilon > 0$ such that, for all $x_0 \in \overline{\Omega}$, $z \in \mathbb{R}^{N \times d}$ and $J \in \mathbb{R}^{d \times d}$, if $w = zJ$ and $|J - I_d| < \delta$, then

$$|G(x_0, 0, w) - G(x_0, 0, z)| < \varepsilon |S(z)|^2.$$

- (b) For every $\varepsilon > 0$ there exist $R = R_\varepsilon > 0$ and $\delta = \delta_\varepsilon > 0$ such that, for all $x_0, x \in \overline{\Omega}$, $z \in \mathbb{R}^{N \times d}$ and $J \in \mathbb{R}^{d \times d}$, if $|x - x_0| < R$, $w = zJ$ and $|y| + |J - I_d| < \delta$, then

$$(5.1) \quad |G(x_0, 0, w) - G(x, y, z)| \det J| < c|y|^2 |\det J| + \varepsilon |S(z)|^2 |\det J|$$

for some constant $c > 0$.

- (c) For every $\varepsilon > 0$ there exists $\delta = \delta_\varepsilon > 0$ such that, for all $z \in \mathbb{R}^{N \times d}$ and $J \in \mathbb{R}^{d \times d}$, if $w = zJ$ and $|J - I_d| < \delta$, then

$$(5.2) \quad c_0 ||S(zJ)|^2 - |S(z)|^2| \det J| < \varepsilon |S(z)|^2.$$

Proof. The proof of (a) is carried out following a similar strategy to the one followed in (b) of Lemma 4.5.

Case 1. If $|z| \leq 1$ and $w = zJ$ with $|J - I_d| < 1$ then, since F_{zz} is locally uniformly continuous, we can find a modulus of continuity, say $\omega: [0, \infty) \rightarrow [0, 1]$, such that it is increasing, continuous, $\omega(0) = 0$ and for which there is a constant $c > 0$ with the property that

$$|F_{zz}(x_0, u_0(x_0), \nabla u_0(x_0) + tz) - F_{zz}(x_0, u_0(x_0), \nabla u_0(x_0) + tw)| \leq c \omega(|z - w|)$$

for all $x_0 \in \overline{\Omega}$, $t \in [0, 1]$, $|z| \leq 1$ and $w = zJ$ with $|J - I_d| < 1$. We can further assume that c is such that

$$(5.3) \quad 1 + d + |F_{zz}(x_0, u_0(x_0), \nabla u_0(x_0) + tz)| \leq c$$

for all $x_0 \in \overline{\Omega}$, $t \in [0, 1]$ and $|z| \leq 1$. Then fixing $0 < \varepsilon < 1$, we find that

$$\begin{aligned} & |G(x_0, 0, w) - G(x_0, 0, z)| \\ & \leq \int_0^1 |F_{zz}(x_0, u_0(x_0), \nabla u_0(x_0) + tw)w \cdot w - F_{zz}(x_0, u_0(x_0), \nabla u_0(x_0) + tz)z \cdot z| dt \\ & \leq \int_0^1 (|F_{zz}(x_0, u_0(x_0), \nabla u_0(x_0) + tw)w \cdot w - F_{zz}(x_0, u_0(x_0), \nabla u_0(x_0) + tz)w \cdot w|) dt \\ & \quad + \int_0^1 |F_{zz}(x_0, u_0(x_0), \nabla u_0(x_0) + tz)w \cdot w - F_{zz}(x_0, u_0(x_0), \nabla u_0(x_0) + tz)z \cdot z| dt \\ & \leq c \omega(|w - z|)|w|^2 + \int_0^1 |F_{zz}(x_0, u_0(x_0), \nabla u_0(x_0) + tz)||w - z|(|w| + |z|) dt \\ & \leq c \omega(|J - I_d|)|J|^2|z|^2 + |F_{zz}(x_0, u_0(x_0), \nabla u_0(x_0) + tz)||J - I_d|(|J||z|^2 + |z|^2) \\ & \leq c \omega(|J - I_d|)|J|^2|z|^2 + c|J - I_d|(|J||z|^2 + |z|^2) \\ & \leq \varepsilon |z|^2, \end{aligned}$$

where the last inequality is making use of (5.3) and holds provided that $|J - I_d|$ and $\omega(|J - I_d|)$ are small enough. Note that, if this is the case, we can assume that $|J| < c(d)$ for a constant $c(d) > 0$. Thus, for the case $|z| \leq 1$ we have the desired inequality if $|J - I_d| < \delta_\varepsilon < 1$.

Case 2. For $|z| > 1$, we make use of [H1] (b) and the fact that F_z is also locally uniformly continuous, to deduce that for $x_0 \in \overline{\Omega}$

$$\begin{aligned} |G(x_0, 0, w) - G(x_0, 0, z)| &\leq |F(x_0, u_0(x_0), \nabla u_0(x_0) + w) - F(x_0, u_0(x_0), \nabla u_0(x_0) + z)| \\ &\quad + |F_z(x_0, u_0(x_0), \nabla u_0(x_0))||w - z| \\ &\leq \tilde{c}_1 (1 + |w|^{p-1} + |z|^{p-1})|w - z| \\ &\leq c(|z| + |z|^p)|J - I_d| \\ &\leq C|J - I_d||z|^p, \end{aligned}$$

where the last inequality follows from the fact that $|z| > 1$. Therefore, if for a given $\varepsilon > 0$ we take $\delta = \frac{\varepsilon}{C}$ and $|J - I_d| < \delta$, then

$$C|J - I_d| < \varepsilon$$

and the claim follows.

The proof of (b) is an easy consequence of Lemma 4.5 (b) and of (a) above. We first use the triangle inequality to estimate

$$\begin{aligned} |G(x_0, 0, w) - G(x, y, z)| &\leq |G(x_0, 0, w) - G(x_0, 0, z)| + |G(x_0, 0, z) - G(x, y, z)| \\ &=: \text{I} + \text{II}. \end{aligned}$$

By part (a) we deduce that $\text{I} < \frac{\varepsilon}{4}(|z|^2 + |z|^p)$ whenever $|J - I_d| < \delta$ for some $\delta \in (0, 1)$ depending solely on ε . Furthermore, we note that, in this case, we may also ensure that $|J| \leq c(d)$ for some constant $c(d) > 0$. This, together with part (b) of Lemma 4.5, imply that $\text{II} < c|y|^2 + \frac{\varepsilon}{4}(|z|^2 + |z|^p)$ whenever $|x - x_0|$ and $|y|$ are small enough. Whereby,

$$|G(x_0, 0, w) - G(x, y, z)| < c|y|^2 + \frac{\varepsilon}{2}(|z|^2 + |z|^p).$$

The conclusion of (b) follows directly from this and the continuity of the determinant, after using the triangle inequality and Lemma 4.5 (a).

Finally, to prove (c) we observe that, since the determinant is a continuous function, for any given $C, \varepsilon > 0$ there is a $\delta = \delta_\varepsilon \in (0, \min\{\frac{\varepsilon}{2}, 1\})$ such that, if $|J - I_d| < \delta$ with $J \in \mathbb{R}^{d \times d}$, we can ensure that $|J - I_d| + |1 - \det(J)| < \frac{\varepsilon}{4C}$. This technical observation enables us to estimate, for any $z \in \mathbb{R}^{N \times d}$ and $J \in \mathbb{R}^{d \times d}$ with $|J - I_d| < \delta$ as above, that

$$\begin{aligned} c_0 ||S(zJ)|^2 - |S(z)|^2| \det J| &\leq c_0 [|S(zJ)|^2 - |S(z)|^2 + |S(z)|^2 |1 - \det J|] \\ (5.4) \quad &\leq C(|zJ| + |z| + |zJ|^{p-1} + |z|^{p-1})|zJ - z| \end{aligned}$$

$$\begin{aligned} (5.5) \quad &\quad + C|S(z)|^2 |1 - \det J| \\ &\leq C[|S(z)|^2 |J - I_d| + |S(z)|^2 |1 - \det J|] \end{aligned}$$

$$(5.6) \quad < \varepsilon |S(z)|^2.$$

We remark that inequality (5.5) follows from the Lipschitz properties of the function S , since $|t^p - s^p| \leq c(p)(t^{p-1} + s^{p-1})|t - s|$ for $t, s \geq 0$, $p \geq 1$. Note also that we are using the inequality $|J| \leq C$, which we may assume since $|J - I_d| < \delta < 1$. This concludes the proof of the Lemma. \square

We now proceed with the proof of Theorem 4.4 in its most general case.

Proof of Theorem 4.4 (general case). The first part of the proof consists in constructing appropriate diffeomorphisms that transform sets of the type $\Omega(x_0, R)$ into sets in which we can apply an

appropriate *quaxiconvexity at the boundary* condition. More precisely, we construct diffeomorphisms that locally “flatten” the facelike parts of the boundary of Ω by smoothly mapping them into the faces of a polytope.

By compactness of $\partial\Omega$, there are a finite set $\{y_1, \dots, y_M\} \subseteq \partial\Omega$ and radii $\bar{r}_1, \dots, \bar{r}_M > 0$ such that

$$\partial\Omega \subseteq \bigcup_{j=1}^M B\left(y_j, \frac{\bar{r}_j}{2}\right)$$

and for which there exist polytopes $\mathcal{P}_1, \dots, \mathcal{P}_M$ and orientation-preserving diffeomorphisms $g_j: \overline{\Omega(y_j, \bar{r}_j)} \rightarrow \mathcal{D}_j \subseteq \mathcal{P}_j$ with $g_j(\partial\Omega \cap \overline{B(y_j, \bar{r}_j)}) \subseteq \partial\mathcal{P}_j$.

On the other hand we observe that, for $x_1 \in \Gamma_N$ arbitrary, $x_1 \in B(y_j, \bar{r}_j)$ for some $1 \leq j \leq M$. We now denote

$$A_1 := \nabla g_j(x_1) \in \mathbb{R}^{d \times d} \quad \text{and} \quad P_1 := A_1^{-1} \mathcal{D}_j - A_1^{-1} g_j(x_1) + x_1 \subseteq \mathbb{R}^d.$$

Consider the diffeomorphism $\Phi_1: P_1 \rightarrow \overline{\Omega(x_j, \bar{r}_j)}$ given by

$$(5.7) \quad \Phi_1(\xi) := g_j^{-1}(A_1(\xi - x_1) + g_j(x_1)).$$

Clearly,

$$\Phi_1^{-1}(x) = A_1^{-1} \cdot (g_j(x) - g_j(x_1)) + x_1$$

and, hence, Φ_1 is well defined. We then observe that, for $\xi \in P_1$,

$$\begin{aligned} & |\Phi_1(\xi) - \xi| + |\nabla \Phi_1(\xi) - \text{Id}| \\ &= |g_j^{-1}(A_1(\xi - x_1) + g_j(x_1)) - g_j^{-1}(g_j(\xi))| \\ &+ \left| \left[\nabla g_j^{-1}(A_1(\xi - x_1) + g_j(x_1)) - \nabla g_j^{-1}(g_j(x_1)) \right] \nabla g_j(x_1) \right| \\ &= |g_j^{-1}(A_1(\xi - x_1) + g_j(x_1)) - g_j^{-1}(g_j(\xi))| \\ &+ \left| \left[\nabla g_j^{-1}(A_1(\xi - x_1) + g_j(x_1)) - \nabla g_j^{-1}(g_j(x_1)) \right] \nabla g_j(x_1) \right| \\ (5.8) \quad &\leq c\omega_0^j(|A_1||\xi - x_1| + |g_j(x_1) - g_j(\xi)|) + c|A_1|\omega_1^j(|A_1||\xi - x_1|), \end{aligned}$$

where ω_0^j and ω_1^j are moduli of continuity of g_j^{-1} and ∇g_j^{-1} respectively.

Since, in particular, g_j is continuous over the compact set $\overline{\Omega(x_j, \bar{r}_j)}$ and the set $\{g_1, \dots, g_M\}$ is finite, this implies that, given $\delta > 0$, there exists some $R_0 > 0$, that does not depend on x_1 , such that

$$(5.9) \quad \|\Phi_1 - \text{Id}_{P_1}\|_{C^1(P_1 \cap B(x_1, R_0), \mathbb{R}^d)} + \|\det \nabla \Phi_1 - 1\|_{C^0(P_1 \cap B(x_1, R_0))} < \delta.$$

By uniform continuity and the fact that $\Phi_1(x_1) = x_1$, we can further find $R_1 > 0$, independent of x_1 , such that

$$(5.10) \quad \Phi_1^{-1}(B(x_1, R_1)) \subseteq B(x_1, R_0).$$

Having established the above estimates, for a fixed $\varepsilon > 0$ we use Lemma 5.1 (b)-(c) to find $0 < \delta < 1$ and $R_\varepsilon > 0$ such that the inequality

$$\begin{aligned} & |G(x_0, 0, zJ) - c_0|S(zJ)|^2 - (G(x, y, z)|\det J| - c_0|S(z)|^2|\det J|)| \\ (5.11) \quad &\leq c|y|^2|\det J| + \varepsilon|S(z)|^2|\det J| \end{aligned}$$

is satisfied whenever $x \in \Omega(x_0, R_\varepsilon)$ and $|J - \text{Id}| < \delta$. For such $\delta > 0$, we may take R_0 such that (5.9) holds. We further assume that $\|\Phi_1\|_{C^1(P_1 \cap B(x_1, R_0), \mathbb{R}^d)} \leq 2$ and we set

$$R := \frac{1}{4} \min\{R_\varepsilon, R_1, \bar{r}_j, \text{diam}(\Omega) : 1 \leq j \leq M\}.$$

In order to show that u_0 indeed satisfies $(*)$ of Theorem 4.4, we must now consider the following cases.

Case 1. If $\overline{\Omega(x_0, R)} \cap \Gamma_N \neq \emptyset$, we choose $x_1 \in \overline{\Omega(x_0, R)} \cap \Gamma_N$ such that x_1 is a $(d-k)$ -facelike point of Ω , where $d-k$ is the minimum dimension of the facelike portions of the boundary that $\overline{\Omega(x_0, R)}$ intersects. Recall that we assume there is precisely one facelike boundary of dimension $d-k$ intersecting $\overline{\Omega(x_0, R)}$. Whereby, $\Omega(x_0, R) \subseteq \Omega(x_1, 2R)$. Furthermore, if $x_1 \in \Omega(y_j, \frac{\bar{r}_j}{2})$, then $\Omega(x_1, 2R) \subseteq \Omega(y_j, \frac{\bar{r}_j}{2})$.

We now consider Φ_1 defined as in (5.7). Then, it follows from (5.10) that, for any $\xi \in \Phi_1^{-1}(\Omega(x_0, R))$, we have $|\xi - x_1| < R_0$ and, hence, if $\varphi \in \text{Var}(\Omega(x_0, R), \mathbb{R}^N)^{W^{1,p}}$ satisfies

$$(5.12) \quad \|\varphi\|_{L^\infty(\Omega(x_0, R), \mathbb{R}^N)} < \delta$$

with $\delta > 0$ as above, we can substitute $x = \Phi_1(\xi)$, $y = \varphi(\Phi_1(\xi))$, $z := \nabla \varphi(\Phi_1(\xi))$ and $J := \nabla \Phi_1(\xi)$ in (5.11) to obtain, after integrating over $\Phi_1^{-1}(\Omega(x_0, R))$,

$$(5.13) \quad \begin{aligned} & \int_{\Phi_1^{-1}(\Omega(x_0, R))} G(x_0, 0, \nabla \varphi(\Phi_1(\xi)) \nabla \Phi_1(\xi)) d\xi \\ & - c_0 \int_{\Phi_1^{-1}(\Omega(x_0, R))} |S(\nabla \varphi(\Phi_1(\xi)) \nabla \Phi_1(\xi))|^2 d\xi \\ & - \int_{\Phi_1^{-1}(\Omega(x_0, R))} G(\Phi_1(\xi), \varphi(\Phi_1(\xi)), \nabla \varphi(\Phi_1(\xi))) |\det \nabla \Phi_1(\xi)| d\xi \\ & + c_0 \int_{\Phi_1^{-1}(\Omega(x_0, R))} |S(\nabla \varphi(\Phi_1(\xi)))|^2 |\det \nabla \Phi_1(\xi)| d\xi \\ & \leq c \int_{\Phi_1^{-1}(\Omega(x_0, R))} |\varphi(\Phi_1(\xi))|^2 |\det \nabla \Phi_1(\xi)| d\xi \\ & + \varepsilon \int_{\Phi_1^{-1}(\Omega(x_0, R))} |S(\nabla \varphi(\Phi_1(\xi)))|^2 |\det \nabla \Phi_1(\xi)| d\xi. \end{aligned}$$

We are interested in using the quasiconvexity at the boundary condition in order to simplify the above expression. With this aim, we define $\tilde{\varphi}: \Phi_1^{-1}(\Omega(x_0, R)) \rightarrow \mathbb{R}^N$ as

$$\tilde{\varphi}(\xi) := \varphi(\Phi_1(\xi)).$$

We wish to establish that $\tilde{\varphi}$ is a suitable test function for the quasiconvexity at the boundary condition that we have imposed. Assuming without loss of generality that $x_1 = g_j(x_1) = 0$, we know that if $m_i(g_j(x_1))$ is a normal vector to a $(d-1)$ -face of the polytopal domain $\mathcal{D}_j := g_j(\overline{\Omega(y_j, \bar{r}_j)})$ on which $g_j(x_1)$ lies, then any point y on that face satisfies

$$m_i(g_j(x_1)) \cdot y = 0.$$

In addition, if $n_i(x_1)$ is the associated normal to Ω at the point x_1 and $A_1 := \nabla g_j(x_1)$, then, by definition of $n_i(x_1)$ we have

$$(5.14) \quad n_i(x_1) \cdot A_1^{-1}y = (\det A_1)^{-1} [A_1^T m_i(g_j(x_1)) \cdot A_1^{-1}y] = (\det A_1)^{-1} [m_i(g_j(x_1)) \cdot y] = 0.$$

This implies that, under the diffeomorphism Φ_1^{-1} restricted to the region $\Omega(x_0, R)$, each $(d-1)$ -facelike portion of the boundary of Ω is mapped into the portion of the hyperplane defined by the normal vector $m_i(g_j(x_1))$ intersected with the region $\Phi_1^{-1}(B(x_0, R))$. Similarly, any outwards, or inwards, pointing $(d-k)$ -facelike boundary is mapped under the diffeomorphism Φ_1^{-1} into the portion of ∂P_1 defined as the intersection of the hyperplanes with normals $m_i(g_j(x_1))$.

Hence, since $\varphi \in \text{Var}(\Omega(x_0, R), \mathbb{R}^N)^{W^{1,p}}$, $\tilde{\varphi}$ is a suitable test function for the quasiconvexity at the free boundary condition, by Remark 2.10 and Lemma 2.12.

Noting that $\nabla \tilde{\varphi}(\xi) = \nabla \varphi(\Phi_1(\xi)) \nabla \Phi_1(\xi)$, we deduce

$$(5.15) \quad \begin{aligned} 0 &\leq \int_{\Phi_1^{-1}(\Omega(x_0, R))} (F(x_0, u_0(x_0), \nabla u_0(x_0) + \nabla \varphi(\Phi_1(\xi)) \nabla \Phi_1(\xi)) - F(x_0)) \, d\xi \\ &\quad - c_0 \int_{\Phi_1^{-1}(\Omega(x_0, R))} |S(\nabla \varphi(\Phi_1(\xi)) \nabla \Phi_1(\xi))|^2 \, d\xi. \end{aligned}$$

Moreover, the weak Euler-Lagrange equations associated to the above minimality condition imply that

$$\int_{\Phi_1^{-1}(\Omega(x_0, R))} F_z(x_0, u_0(x_0), \nabla u_0(x_0)) \cdot (\nabla \varphi(\Phi_1(\xi)) \nabla \Phi_1(\xi)) \, d\xi = 0.$$

Whereby, (5.15) now reads

$$(5.16) \quad 0 \leq \int_{\Phi_1^{-1}(\Omega(x_0, R))} G(x_0, 0, \nabla(\varphi \circ \Phi_1)(\xi)) \, d\xi - c_0 \int_{\Phi_1^{-1}(\Omega(x_0, R))} |S(\nabla(\varphi \circ \Phi_1)(\xi))|^2 \, d\xi.$$

From inequalities (5.13) and (5.16), we infer that

$$\begin{aligned} & - \int_{\Phi_1^{-1}(\Omega(x_0, R))} G(\Phi_1(\xi), \varphi \circ \Phi_1(\xi), \nabla \varphi(\Phi_1(\xi)) |\det \nabla \Phi_1(\xi)| \, d\xi \\ & + c_0 \int_{\Phi_1^{-1}(\Omega(x_0, R))} |S(\nabla \varphi(\Phi_1(\xi)))|^2 |\det \nabla \Phi_1(\xi)| \, d\xi \\ & \leq \varepsilon \int_{\Phi_1^{-1}(\Omega(x_0, R))} |S(\nabla \varphi(\Phi_1(\xi)))|^2 |\det \nabla \Phi_1(\xi)| \, d\xi \\ & + c \int_{\Phi_1^{-1}(\Omega(x_0, R))} |\varphi(\Phi_1(\xi))|^2 |\det \nabla \Phi_1(\xi)| \, d\xi. \end{aligned}$$

Applying the change of variables $x = \Phi_1(\xi)$, this leads to

$$(5.17) \quad \begin{aligned} & - \int_{\Omega(x_0, R)} G(x, \varphi(x), \nabla \varphi(x)) + c_0 |S(\nabla \varphi(x))|^2 \, dx \\ & \leq \varepsilon \int_{\Omega(x_0, R)} |S(\nabla \varphi(x))|^2 \, dx + c \int_{\Omega(x_0, R)} |\varphi(x)|^2 \, dx. \end{aligned}$$

Also, since $\varphi \in \overline{\text{Var}(\Omega(x_0, R), \mathbb{R}^N)^{W^{1,p}}}$ and u_0 is an F -extremal,

$$\int_{\Omega(x_0, R)} F_y(x, u_0(x), \nabla u_0(x)) \cdot \varphi(x) + F_z(x, u_0(x), \nabla u_0(x)) \cdot \nabla \varphi(x) \, dx = 0.$$

This, together with (5.17), imply for $\varepsilon = \frac{c_0}{2}$ that

$$\begin{aligned} & \int_{\Omega(x_0, R)} (F(x, u_0(x) + \varphi(x), \nabla u_0(x) + \nabla \varphi(x)) - F(x, u_0(x), \nabla u_0(x))) \, dx \\ & - c_0 \int_{\Omega(x_0, R)} |S(\nabla \varphi(x))|^2 \, dx + c \int_{\Omega(x_0, R)} |\varphi(x)|^2 \, dx \\ & \geq - \frac{c_0}{2} \int_{\Omega(x_0, R)} |S(\nabla \varphi(x))|^2 \, dx, \end{aligned}$$

which gives the desired inequality after adding $c_0 \int_{\Omega(x_0, R)} |S(\nabla \varphi(x))|^2 \, dx$ to both sides of the above expression. This concludes the proof of (*) for x_0 in a neighbourhood of Γ_N .

Case 2. To prove (*) if $x_0 \in \overline{\Omega}$ is not in the neighbourhood of radius R of Γ_N , a simpler version of the above proof works, since we can then use that the standard quasiconvexity holds in $\overline{\Omega}$ and take Φ_1 as the identity diffeomorphism in the above proof, given that there is no need, in this case, to transform the boundary into a subset of a polytope. All other calculations follow in the exact same way. This concludes the proof of the theorem. \square

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REFERENCES

- [ADMD12] V. Agostiniani, G. Dal Maso, and A. DeSimone. Linear elasticity obtained from finite elasticity by Γ -convergence under weak coerciveness conditions. *Annales Inst. Henri Poincaré (C) Non Linear Analysis*, 29(5):715–735, 2012.
- [AF87] E. Acerbi and N. Fusco. A regularity theorem for minimizers of quasiconvex integrals. *Arch. Rational Mech. Anal.*, 99(3):261–281, 1987.
- [Bal98] J. M. Ball. The calculus of variations and materials science. *Quart. Appl. Math.*, 56(4):719–740, 1998. Current and future challenges in the applications of mathematics (Providence, RI, 1997).
- [BK16] J. M. Ball and K. Koumatos. Quasiconvexity at the boundary and the nucleation of austenite. *Arch. Rational Mech. Anal.*, 219(1):89–157, 2016.
- [BKS13] J. M. Ball, K. Koumatos, and H. Seiner. Nucleation of austenite in mechanically stabilized martensite by localized heating. *Journal of Alloys and Compounds*, 577:S37–S42, 2013.
- [BM84a] J. M. Ball and J. E. Marsden. Quasiconvexity at the boundary, positivity of the second variation and elastic stability. *Arch. Rational Mech. Anal.*, 86(3):251–277, 1984.
- [BM84b] J. M. Ball and F. Murat. $W^{1,p}$ -quasiconvexity and variational problems for multiple integrals. *J. Funct. Anal.*, 58(3):225–253, 1984.
- [CC] J. Campos Cordero. Boundary regularity and sufficient conditions for strong local minimizers. Preprint.
- [Dac08] B. Dacorogna. *Direct methods in the calculus of variations*, volume 78 of *Applied Mathematical Sciences*. Springer, New York, second edition, 2008.
- [Eva86] L. C. Evans. Quasiconvexity and partial regularity in the calculus of variations. *Arch. Rational Mech. Anal.*, 95(3):227–252, 1986.
- [FMP98] I. Fonseca, S. Müller, and P. Pedregal. Analysis of concentration and oscillation effects generated by gradients. *SIAM J. Math. Anal.*, 29(3):736–756 (electronic), 1998.
- [Giu03] E. Giusti. *Direct methods in the calculus of variations*. World Scientific Publishing Co. Inc., River Edge, NJ, 2003.
- [GM09] Y. Grabovsky and T. Mengesha. Sufficient conditions for strong local minimizers: the case of C^1 extremals. *Trans. Amer. Math. Soc.*, 361(3):1495–1541, 2009.
- [Grü03] B. Grünbaum. Convex polytopes, Volume 221 of Graduate Texts in Mathematics, 2003.
- [Hes48] M. R. Hestenes. Sufficient conditions for multiple integral problems in the calculus of variations. *Amer. J. Math.*, 70:239–276, 1948.
- [KKK14] A. Kałamajska, S. Krömer, and M. Kružík. Sequential weak continuity of null Lagrangians at the boundary. *Calculus of Variations and Partial Differential Equations*, 49(3-4):1263–1278, 2014.
- [Kri94] J. Kristensen. Finite functionals and Young measures generated by gradients of Sobolev functions. *Technical Report Mat-Report No. 1994-34, Mathematical Institute, Technical University of Denmark*, 1994.
- [Kri99] J. Kristensen. Lower semicontinuity in spaces of weakly differentiable functions. *Math. Ann.*, 313(4):653–710, 1999.
- [Kru13] M. Kružík. Quasiconvexity at the boundary and concentration effects generated by gradients. *ESAIM: Control, Optimisation and Calculus of Variations*, 19(03):679–700, 2013.
- [KT03] J. Kristensen and A. Taheri. Partial regularity of strong local minimizers in the multi-dimensional calculus of variations. *Arch. Ration. Mech. Anal.*, 170(1):63–89, 2003.
- [Mey65] N. G. Meyers. Quasi-convexity and lower semi-continuity of multiple variational integrals of any order. *Trans. Amer. Math. Soc.*, 119:125–149, 1965.
- [MS98] A. Mielke and P. Sprenger. Quasiconvexity at the boundary and a simple variational formulation of Agmon’s condition. *Journal of Elasticity*, 51(1):23–41, 1998.
- [MŠ03] S. Müller and V. Šverák. Convex integration for Lipschitz mappings and counterexamples to regularity. *Ann. of Math. (2)*, 157(3):715–742, 2003.
- [Szé04] L. Székelyhidi, Jr. The regularity of critical points of polyconvex functionals. *Arch. Ration. Mech. Anal.*, 172(1):133–152, 2004.
- [Tah01] A. Taheri. Sufficiency theorems for local minimizers of the multiple integrals of the calculus of variations. *Proc. Roy. Soc. Edinburgh Sect. A*, 131(1):155–184, 2001.

- [Zha92] K. Zhang. Remarks on quasiconvexity and stability of equilibria for variational integrals. *Proceedings of the American Mathematical Society*, 114(4):927–930, 1992.

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